NAG Library Routine Document

G01HBF

Note: before using this routine, please read the Users' Note for your implementation to check the interpretation of ***bold italicised*** terms and other implementation-dependent details.

1 Purpose

G01HBF returns the upper tail, lower tail or central probability associated with a multivariate Normal distribution of up to ten dimensions.

2 Specification

```
double precision FUNCTION G01HBF(TAIL, N, A, B, XMU, SIG, LDSIG, TOL, WK,
1                                LWK, IFAIL)
INTEGER                         N, LDSIG, LWK, IFAIL
double precision                A(N), B(N), XMU(N), SIG(LDSIG,N), TOL,
1                               WK(LWK)
CHARACTER*1                     TAIL
```

3 Description

Let the vector random variable $X = (X_1, X_2, \ldots, X_n)^{\mathrm{T}}$ follow an n-dimensional multivariate Normal distribution with mean vector μ and n by n variance-covariance matrix Σ, then the probability density function, $f(X : \mu, \Sigma)$, is given by

$$f(X : \mu, \Sigma) = (2\pi)^{-(1/2)n} |\Sigma|^{-1/2} \exp\left(-\tfrac{1}{2}(X - \mu)^{\mathrm{T}} \Sigma^{-1} (X - \mu)\right).$$

The lower tail probability is defined by:

$$P(X_1 \leq b_1, \ldots, X_n \leq b_n : \mu, \Sigma) = \int_{-\infty}^{b_1} \cdots \int_{-\infty}^{b_n} f(X : \mu, \Sigma)\, dX_n \cdots dX_1.$$

The upper tail probability is defined by:

$$P(X_1 \geq a_1, \ldots, X_n \geq a_n : \mu, \Sigma) = \int_{a_1}^{\infty} \cdots \int_{a_n}^{\infty} f(X : \mu, \Sigma)\, dX_n \cdots dX_1.$$

The central probability is defined by:

$$P(a_1 \leq X_1 \leq b_1, \ldots, a_n \leq X_n \leq b_n : \mu, \Sigma) = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(X : \mu, \Sigma)\, dX_n \cdots dX_1.$$

To evaluate the probability for $n \geq 3$, the probability density function of $X_1, X_2, \ldots, X_n$ is considered as the product of the conditional probability of $X_1, X_2, \ldots, X_{n-2}$ given X_{n-1} and X_n and the marginal bivariate Normal distribution of X_{n-1} and X_n. The bivariate Normal probability can be evaluated as described in G01HAF and numerical integration is then used over the remaining $n - 2$ dimensions. In the case of $n = 3$, D01AJF is used and for $n > 3$ D01FCF is used.

To evaluate the probability for $n = 1$ a direct call to G01EAF is made and for $n = 2$ calls to G01HAF are made.

4 References

Kendall M G and Stuart A (1969) *The Advanced Theory of Statistics (Volume 1)* (3rd Edition) Griffin

5 Parameters

1: TAIL – CHARACTER*1 *Input*

On entry: indicates which probability is to be returned.

TAIL = 'L'
The lower tail probability is returned.

TAIL = 'U'
The upper tail probability is returned.

TAIL = 'C'
The central probability is returned.

Constraint: TAIL = 'L', 'U' or 'C'.

2: N – INTEGER *Input*

On entry: n, the number of dimensions.

Constraint: $1 \leq \text{N} \leq 10$.

3: A(N) – ***double precision*** array *Input*

On entry: if TAIL = 'C' or 'U', the lower bounds, a_i, for $i = 1, 2, \ldots, n$.

If TAIL = 'L', A is not referenced.

4: B(N) – ***double precision*** array *Input*

On entry: if TAIL = 'C' or 'L', the upper bounds, b_i, for $i = 1, 2, \ldots, n$.

If TAIL = 'U' B, is not referenced.

Constraint: if TAIL = 'C', $\text{A}(i) < \text{B}(i)$ for $i = 1, 2, \ldots, n$.

5: XMU(N) – ***double precision*** array *Input*

On entry: μ, the mean vector of the multivariate Normal distribution.

6: SIG(LDSIG,N) – ***double precision*** array *Input*

On entry: Σ, the variance-covariance matrix of the multivariate Normal distribution. Only the lower triangle is referenced.

Constraint: Σ must be positive-definite

7: LDSIG – INTEGER *Input*

On entry: the first dimension of the array SIG as declared in the (sub)program from which G01HBF is called.

Constraint: $\text{LDSIG} \geq \text{N}$.

8: TOL – ***double precision*** *Input*

On entry: if $n > 2$ the relative accuracy required for the probability, and if the upper or the lower tail probability is requested then TOL is also used to determine the cut-off points, see Section 7.

If $n = 1$, TOL is not referenced.

Suggested value: TOL = 0.0001.

Constraint: if $\text{N} > 1$, $\text{TOL} > 0.0$.

9: WK(LWK) – ***double precision*** array *Workspace*

10: LWK – INTEGER *Input*

On entry: the length of workspace provided in array WK. This workspace is used by the numerical integration routines D01AJF for $n = 3$ and D01FCF for $n > 3$.

If $n = 3$, then the maximum number of sub-intervals used by D01AJF is LWK/4. Note, however, increasing LWK above 1000 will not increase the maximum number of sub-intervals above 250.

If $n > 3$ the maximum number of integrand evaluations used by D01FCF is $\alpha(\mathrm{LWK}/n - 1)$, where $\alpha = 2^{n-2} + 2(n-2)^2 + 2(n-2) + 1$.

If $n = 1$ or 2, then WK will not be used.

Suggested value: 2000 if $n > 3$ and 1000 if $n = 3$.

Constraints:

if $\mathrm{N} = 1$, $\mathrm{LWK} \geq 1$;
if $\mathrm{N} \geq 3$, $\mathrm{LWK} \geq 4 \times \mathrm{N}$.

11: IFAIL – INTEGER *Input/Output*

On entry: IFAIL must be set to 0, -1 or 1. If you are unfamiliar with this parameter you should refer to Section 3.3 in the Essential Introduction for details.

On exit: IFAIL $= 0$ unless the routine detects an error (see Section 6).

For environments where it might be inappropriate to halt program execution when an error is detected, the value -1 or 1 is recommended. If the output of error messages is undesirable, then the value 1 is recommended. Otherwise, because for this routine the values of the output parameters may be useful even if IFAIL $\neq 0$ on exit, the recommended value is -1. **When the value -1 or 1 is used it is essential to test the value of IFAIL on exit.**

6 Error Indicators and Warnings

If on entry IFAIL $= 0$ or -1, explanatory error messages are output on the current error message unit (as defined by X04AAF).

Note: G01HBF may return useful information for one or more of the following detected errors or warnings.

Errors or warnings detected by the routine:

If on exit IFAIL $= 1$, 2 or 3, then G01HBF returns zero.

IFAIL $= 1$

On entry, $\mathrm{N} < 1$,
or $\mathrm{N} > 10$,
or $\mathrm{LDSIG} < \mathrm{N}$,
or TAIL $\neq$ 'L', 'U' or 'C',
or $\mathrm{N} > 1$ and $\mathrm{TOL} \leq 0.0$,
or LWK is too small.

IFAIL $= 2$

On entry, TAIL = 'C' and $\mathrm{A}(i) \geq \mathrm{B}(i)$, for some $i = 1, 2, \ldots, n$.

IFAIL $= 3$

On entry, Σ is not positive-definite, i.e., is not a correct variance-covariance matrix.

IFAIL = 4

The requested accuracy has not been achieved, a larger value of TOL should be tried or the length of the workspace should be increased. The returned value will be an approximation to the required result.

IFAIL = 5

Round-off error prevents the requested accuracy from being achieved; a larger value of TOL should be tried. The returned value will be an approximation to the required result. This error will only occur if $n = 3$.

7 Accuracy

The accuracy should be as specified by TOL. When on exit IFAIL = 4 the approximate accuracy achieved is given in the error message. For the upper and lower tail probabilities the infinite limits are approximated by cut-off points for the $n - 2$ dimensions over which the numerical integration takes place; these cut-off points are given by $\Phi^{-1}(\mathrm{TOL}/(10 \times n))$, where Φ^{-1} is the inverse univariate Normal distribution function.

8 Further Comments

The time taken is related to the number of dimensions, the range over which the integration takes place ($b_i - a_i$, for $i = 1, 2, \ldots, n$) and the value of Σ as well as the accuracy required. As the numerical integration does not take place over the last two dimensions speed may be improved by arranging X so that the largest ranges of integration are for X_{n-1} and X_n.

9 Example

This example reads in the mean and covariance matrix for a multivariate Normal distribution and computes and prints the associated central probability.

9.1 Program Text

```
*     G01HBF Example Program Text
*     Mark 15 Release. NAG Copyright 1991.
*     .. Parameters ..
      INTEGER          NIN, NOUT
      PARAMETER        (NIN=5,NOUT=6)
      INTEGER          NMAX, LWK
      PARAMETER        (NMAX=10,LWK=2000)
*     .. Local Scalars ..
      DOUBLE PRECISION PROB, TOL
      INTEGER          I, IFAIL, J, LDSIG, N
      CHARACTER        TAIL
*     .. Local Arrays ..
      DOUBLE PRECISION A(NMAX), B(NMAX), SIG(NMAX,NMAX), WK(LWK),
     +                 XMU(NMAX)
*     .. External Functions ..
      DOUBLE PRECISION G01HBF
      EXTERNAL         G01HBF
*     .. Executable Statements ..
      WRITE (NOUT,*) 'G01HBF Example Program Results'
*     Skip heading in data file
      READ (NIN,*)
      READ (NIN,*) N, TOL, TAIL
      IF (N.LE.NMAX) THEN
         READ (NIN,*) (XMU(J),J=1,N)
         DO 20 I = 1, N
            READ (NIN,*) (SIG(I,J),J=1,N)
   20    CONTINUE
         IF (TAIL.EQ.'C' .OR. TAIL.EQ.'c' .OR. TAIL.EQ.'U' .OR. TAIL.EQ.
     +       'u') READ (NIN,*) (A(J),J=1,N)
         IF (TAIL.EQ.'C' .OR. TAIL.EQ.'c' .OR. TAIL.EQ.'L' .OR. TAIL.EQ.
     +       'l') READ (NIN,*) (B(J),J=1,N)
```

```
         LDSIG = NMAX
         IFAIL = 1
*
         PROB = G01HBF(TAIL,N,A,B,XMU,SIG,LDSIG,TOL,WK,LWK,IFAIL)
*
         IF (IFAIL.EQ.0 .OR. IFAIL.GT.3) THEN
            WRITE (NOUT,*)
            WRITE (NOUT,99999) 'Multivariate Normal probability = ',
     +        PROB
         ELSE
            WRITE (NOUT,*)
            WRITE (NOUT,99998) ' ** G01HBF returned with IFAIL = ',
     +        IFAIL
         END IF
      END IF
*
99999 FORMAT (1X,A,F6.4)
99998 FORMAT (1X,A,I5)
      END
```

9.2 Program Data

```
G01HBF Example Program Data
 4  0.0001 'c'
 0.0  0.0  0.0  0.0
 1.0  0.9  0.9  0.9
 0.9  1.0  0.9  0.9
 0.9  0.9  1.0  0.9
 0.9  0.9  0.9  1.0
-2.0 -2.0 -2.0 -2.0
 2.0  2.0  2.0  2.0
```

9.3 Program Results

```
 G01HBF Example Program Results

 Multivariate Normal probability = 0.9142
```