1 Purpose

`s17ad` returns the value of the Bessel Function $Y_1(x)$, via the function name.

2 Syntax

```matlab
[result, ifail] = s17ad(x)
```

3 Description

`s17ad` evaluates an approximation to the Bessel Function of the second kind $Y_1(x)$.

**Note:** $Y_1(x)$ is undefined for $x \leq 0$ and the function will fail for such arguments.

The function is based on four Chebyshev expansions:

For $0 < x \leq 8$,

$$Y_1(x) = \frac{2}{\pi} \ln x \sum_{r=0}^{\infty} a_r T_r(t) - \frac{2}{\pi x} \sum_{r=0}^{\infty} b_r T_r(t), \quad \text{with } t = 2\left(\frac{x}{8}\right)^2 - 1.$$

For $x > 8$,

$$Y_1(x) = \sqrt{\frac{2}{\pi x}} \left\{ P_1(x) \sin \left( x - \frac{3\pi}{4} \right) + Q_1(x) \cos \left( x - \frac{3\pi}{4} \right) \right\},$$

where $P_1(x) = \sum_{r=0}^{\infty} c_r T_r(t)$,

and $Q_1(x) = \frac{8}{x} \sum_{r=0}^{\infty} d_r T_r(t)$, with $t = 2\left(\frac{8}{x}\right)^2 - 1$.

For $x$ near zero, $Y_1(x) \approx -\frac{2}{\pi x}$. This approximation is used when $x$ is sufficiently small for the result to be correct to *machine precision*. For extremely small $x$, there is a danger of overflow in calculating $-\frac{2}{\pi x}$ and for such arguments the function will fail.

For very large $x$, it becomes impossible to provide results with any reasonable accuracy (see Section 7), hence the function fails. Such arguments contain insufficient information to determine the phase of oscillation of $Y_1(x)$; only the amplitude, $\sqrt{\frac{2}{\pi x}}$, can be determined and this is returned on soft failure. The range for which this occurs is roughly related to *machine precision*; the function will fail if $x \gtrsim 1/machine \ precision$.

4 References


Clenshaw C W (1962) Chebyshev Series for Mathematical Functions *Mathematical tables* HMSO
5 Parameters

5.1 Compulsory Input Parameters

1: \( x \) – double scalar
   The argument \( x \) of the function.
   Constraint: \( x > 0.0 \).

5.2 Optional Input Parameters

None.

5.3 Input Parameters Omitted from the MATLAB Interface

None.

5.4 Output Parameters

1: \( \text{result} \) – double scalar
   The result of the function.

2: \( \text{ifail} \) – int32 scalar
   \( \text{ifail} = 0 \) unless the function detects an error (see Section 6).

6 Error Indicators and Warnings

Errors or warnings detected by the function:

\( \text{ifail} = 1 \)

\( x \) is too large. On soft failure the function returns the amplitude of the \( Y_1 \) oscillation, \( \frac{\sqrt{2}}{\pi x} \).

\( \text{ifail} = 2 \)

\( x \leq 0.0 \), \( Y_1 \) is undefined. On soft failure the function returns zero.

\( \text{ifail} = 3 \)

\( x \) is too close to zero, there is a danger of overflow. On soft failure, the function returns the value of \( Y_1(x) \) at the smallest valid argument.

7 Accuracy

Let \( \delta \) be the relative error in the argument and \( E \) be the absolute error in the result. (Since \( Y_1(x) \) oscillates about zero, absolute error and not relative error is significant, except for very small \( x \).)

If \( \delta \) is somewhat larger than the \textit{machine precision} (e.g., if \( \delta \) is due to data errors etc.), then \( E \) and \( \delta \) are approximately related by:

\[ E \approx |xY_0(x) - Y_1(x)|\delta \]

(provided \( E \) is also within machine bounds). Figure 1 displays the behaviour of the amplification factor \( |xY_0(x) - Y_1(x)| \).

However, if \( \delta \) is of the same order as \textit{machine precision}, then rounding errors could make \( E \) slightly larger than the above relation predicts.

For very small \( x \), absolute error becomes large, but the relative error in the result is of the same order as \( \delta \).
For very large $x$, the above relation ceases to apply. In this region, $Y_1(x) \simeq \frac{2}{\pi x} \sin \left( x - \frac{3\pi}{4} \right)$. The amplitude $\frac{2}{\pi x}$ can be calculated with reasonable accuracy for all $x$, but $\sin \left( x - \frac{3\pi}{4} \right)$ cannot. If $x - \frac{3\pi}{4}$ is written as $2N\pi + \theta$ where $N$ is an integer and $0 \leq \theta < 2\pi$, then $\sin \left( x - \frac{3\pi}{4} \right)$ is determined by $\theta$ only. If $x > \delta^{-1}$, $\theta$ cannot be determined with any accuracy at all. Thus if $x$ is greater than, or of the order of, the inverse of the machine precision, it is impossible to calculate the phase of $Y_1(x)$ and the function must fail.

![Figure 1](image)

### 8 Further Comments

None.

### 9 Example

```matlab
x = 1;
[result, ifail] = s17ad(x)
```

result =

-0.7812

ifail =

0