NAG Toolbox for Matlab

\textbf{s21bg}

1 Purpose

s21bg returns a value of the classical (Legendre) form of the incomplete elliptic integral of the third kind, via the function name.

2 Syntax

\[
\text{[result, ifail] = s21bg(dn, phi, dm)}
\]

3 Description

s21bg calculates an approximation to the integral

\[
\Pi(n; \phi \mid m) = \int_0^\phi \left(1 - n \sin^2 \theta\right)^{-1} \left(1 - m \sin^2 \theta\right)^{-1} d\theta,
\]

where \(0 \leq \phi \leq \frac{\pi}{2}\), \(m \sin^2 \phi \leq 1\), \(m\) and \(\sin \phi\) may not both equal one, and \(n \sin^2 \phi \neq 1\).

The integral is computed using the symmetrised elliptic integrals of Carlson (Carlson (1979) and Carlson (1988)). The relevant identity is

\[
\Pi(n; \phi \mid m) = \sin \phi R_F(q, r, 1) + \frac{1}{3} n \sin^3 \phi R_J(q, r, 1, s),
\]

where \(q = \cos^2 \phi\), \(r = 1 - m \sin^2 \phi\), \(s = 1 - n \sin^2 \phi\), \(R_F\) is the Carlson symmetrised incomplete elliptic integral of the first kind (see s21bb) and \(R_J\) is the Carlson symmetrised incomplete elliptic integral of the third kind (see s21bd).

4 References


Carlson B C (1979) Computing elliptic integrals by duplication \textit{Numerische Mathematik} 33 1–16


5 Parameters

5.1 Compulsory Input Parameters

1: \(\text{dn} \quad \text{double scalar}\)
2: \(\text{phi} \quad \text{double scalar}\)
3: \(\text{dm} \quad \text{double scalar}\)

The arguments \(n\), \(\phi\) and \(m\) of the function.

\textit{Constraints:}

\[
0.0 \leq \text{phi} \leq \frac{\pi}{2};
\]
\[
\text{dm} \times \sin^2(\text{phi}) \leq 1.0;
\]

Only one of \(\sin(\text{phi})\) and \(\text{dm}\) may be 1.0;

\[
\text{dn} \times \sin^2(\text{phi}) \neq 1.0.
\]

Note that \(\text{dm} \times \sin^2(\text{phi}) = 1.0\) is allowable, as long as \(\text{dm} \neq 1.0\).
5.2 Optional Input Parameters

None.

5.3 Input Parameters Omitted from the MATLAB Interface

None.

5.4 Output Parameters

1: result – double scalar
   The result of the function.

2: ifail – int32 scalar
   ifail = 0 unless the function detects an error (see Section 6).

6 Error Indicators and Warnings

Errors or warnings detected by the function:

ifail = 1
   phi lies outside the range $[0, \tfrac{\pi}{2}]$. On soft failure, the function returns zero.

ifail = 2
   On entry, $\text{dm} \times \sin^2(\text{phi}) > 1.0$; the function is undefined. On soft failure, the function returns zero.

ifail = 3
   On entry, $\sin(\text{phi}) = 1.0$ and $\text{dm} = 1.0$; the function is infinite. On soft failure, the function returns the largest machine number (see x02al).

ifail = 4
   On entry, the product $\text{dn} \times \sin^2(\text{phi}) = 1.0$; the function is infinite. On soft failure, the function returns the largest machine number (see x02al).

7 Accuracy

In principle s21bg is capable of producing full machine precision. However round-off errors in internal arithmetic will result in slight loss of accuracy. This loss should never be excessive as the algorithm does not involve any significant amplification of round-off error. It is reasonable to assume that the result is accurate to within a small multiple of the machine precision.

8 Further Comments

You should consult the S Chapter Introduction, which shows the relationship between this function and the Carlson definitions of the elliptic integrals. In particular, the relationship between the argument-constraints for both forms becomes clear.

For more information on the algorithms used to compute $R_F$ and $R_J$, see the function documents for s21bb and s21bd, respectively.

If you wish to input a value of phi outside the range allowed by this function you should refer to Section 17.4 of Abramowitz and Stegun (1972) for useful identities.
**Example**

```matlab
result = zeros(3, 1);
ifail = zeros(3, 1, 'int32');
fprintf('
 dn phi dm s21bg ifail
');
for ix = 1:3
    phi = ix*pi/6;
    dm = ix/4;
    dn = ((-1)^(ix+1))*ix*0.1;
    [result(ix), ifail(ix)] = s21bg(dn, phi, dm);
    fprintf(' %7.2f %7.2f %7.2f %12.4f %d
', dn, phi, dm, result(ix), ifail(ix));
end
```

<table>
<thead>
<tr>
<th>dn</th>
<th>phi</th>
<th>dm</th>
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<th>ifail</th>
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