NAG Toolbox for Matlab

s21cc

1 Purpose
s21cc returns the value of one of the Jacobian theta functions \( \theta_0(x, q) \), \( \theta_1(x, q) \), \( \theta_2(x, q) \), \( \theta_3(x, q) \) or \( \theta_4(x, q) \) for a real argument \( x \) and non-negative \( q < 1 \), via the function name.

2 Syntax
[\text{result, ifail}] = s21cc(k, x, q)

3 Description
s21cc evaluates an approximation to the Jacobian theta functions \( \theta_0(x, q) \), \( \theta_1(x, q) \), \( \theta_2(x, q) \), \( \theta_3(x, q) \) and \( \theta_4(x, q) \) given by

\[
\begin{align*}
\theta_0(x, q) &= 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^n \cos(2n\pi x), \\
\theta_1(x, q) &= 2 \sum_{n=0}^{\infty} (-1)^n q^{(n+\frac{1}{2})^2} \sin((2n+1)\pi x), \\
\theta_2(x, q) &= 2 \sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2} \cos((2n+1)\pi x), \\
\theta_3(x, q) &= 1 + 2 \sum_{n=1}^{\infty} q^n \cos(2n\pi x), \\
\theta_4(x, q) &= \theta_0(x, q),
\end{align*}
\]

where \( x \) and \( q \) (the nome) are real with \( 0 \leq q < 1 \).

These functions are important in practice because every one of the Jacobian elliptic functions (see s21cb) can be expressed as the ratio of two Jacobian theta functions (see Whittaker and Watson (1990)). There is also a bewildering variety of notations used in the literature to define them. Some authors (e.g., Section 16.27 of Abramowitz and Stegun (1972)) define the argument in the trigonometric terms to be \( x \) instead of \( \pi x \). This can often lead to confusion, so great care must therefore be exercised when consulting the literature. Further details (including various relations and identities) can be found in the references.

s21cc is based on a truncated series approach. If \( t \) differs from \( x \) or \( -x \) by an integer when \( 0 \leq t \leq \frac{1}{2} \), it follows from the periodicity and symmetry properties of the functions that \( \theta_1(x, q) = \pm \theta_1(t, q) \) and \( \theta_3(x, q) = \pm \theta_3(t, q) \). In a region for which the approximation is sufficiently accurate, \( \theta_1 \) is set equal to the first term \( (n = 0) \) of the transformed series

\[
\theta_1(t, q) = 2 \sqrt{\frac{\pi}{\lambda}} e^{-\lambda t} \sum_{n=0}^{\infty} (-1)^n e^{-\lambda(n+\frac{1}{2})^2} \sin((2n+1)\lambda t)
\]

and \( \theta_3 \) is set equal to the first two terms (i.e., \( n \leq 1 \)) of

\[
\theta_3(t, q) = \sqrt{\frac{\pi}{\lambda}} e^{-\lambda t} \left\{ 1 + 2 \sum_{n=1}^{\infty} e^{-\lambda n^2} \cosh(2n\lambda t) \right\},
\]

where \( \lambda = \pi^2/|\log q| \). Otherwise, the trigonometric series for \( \theta_1(t, q) \) and \( \theta_3(t, q) \) are used. For all values of \( x \), \( \theta_0 \) and \( \theta_2 \) are computed from the relations \( \theta_0(x, q) = \theta_3 \left( \frac{1}{2} - |x|, q \right) \) and \( \theta_2(x, q) = \theta_1 \left( \frac{1}{2} - |x|, q \right) \).
4 References
Tolke F (1966) Praktische Funktionenlehre (Bd. II) 1–38 Springer–Verlag

5 Parameters
5.1 Compulsory Input Parameters
1: k – int32 scalar
   The function $\theta_{k(x,q)}$ to be evaluated. Note that $k = 4$ is equivalent to $k = 0$.
   Constraint: $0 \leq k \leq 4$.
2: x – double scalar
   The argument $x$ of the function.
3: q – double scalar
   The argument $q$ of the function.
   Constraint: $0.0 \leq q < 1.0$.

5.2 Optional Input Parameters
None.

5.3 Input Parameters Omitted from the MATLAB Interface
None.

5.4 Output Parameters
1: result – double scalar
   The result of the function.
2: ifail – int32 scalar
   ifail = 0 unless the function detects an error (see Section 6).

6 Error Indicators and Warnings
Errors or warnings detected by the function:

ifail = 1
   On entry, $k < 0$,
   or $k > 4$,
   or $q < 0.0$,
   or $q \geq 1.0$, 
ifail = 2

   The evaluation has been abandoned because the function value is infinite. The result is returned as
   the largest machine representable number (see x02al).

7    Accuracy

    In principle the function is capable of achieving full relative precision in the computed values. However,
    the accuracy obtainable in practice depends on the accuracy of the standard elementary functions such as
    SIN and COS.

8    Further Comments

    None.

9    Example

    k = int32(2);
    x = 0.7;
    q = 0.4;
    [result, ifail] = s21cc(k, x, q)

    result =
    -0.6929
    ifail =
    0