

## NAG Library Function Document

### nag\_ode\_bvp\_fd\_lin\_gen (d02gbc)

#### 1 Purpose

nag\_ode\_bvp\_fd\_lin\_gen (d02gbc) solves a general linear two-point boundary value problem for a system of ordinary differential equations using a deferred correction technique.

#### 2 Specification

```
#include <nag.h>
#include <nagd02.h>

void nag_ode_bvp_fd_lin_gen (Integer neq,
    void (*fcnf)(Integer neq, double x, double f[], Nag_User *comm),
    void (*fcng)(Integer neq, double x, double g[], Nag_User *comm),
    double a, double b, double c[], double d[], double gam[], Integer mnp,
    Integer *np, double x[], double y[], double tol, Nag_User *comm,
    NagError *fail)
```

#### 3 Description

nag\_ode\_bvp\_fd\_lin\_gen (d02gbc) solves the linear two-point boundary value problem for a system of **neq** ordinary differential equations in the interval  $[a, b]$ . The system is written in the form

$$y' = F(x)y + g(x) \quad (1)$$

and the boundary conditions are written in the form

$$Cy(a) + Dy(b) = \gamma \quad (2)$$

Here  $F(x)$ ,  $C$  and  $D$  are **neq** by **neq** matrices, and  $g(x)$  and  $\gamma$  are **neq** component vectors. The approximate solution to (1) and (2) is found using a finite-difference method with deferred correction. The algorithm is a specialisation of that used in the function nag\_ode\_bvp\_fd\_nonlin\_gen (d02rac) which solves a nonlinear version of (1) and (2). The nonlinear version of the algorithm is described fully in Pereyra (1979).

You need to supply an absolute error tolerance and may also supply an initial mesh for the construction of the finite-difference equations (alternatively a default mesh is used). The algorithm constructs a solution on a mesh defined by adding points to the initial mesh. This solution is chosen so that the error is everywhere less than your tolerance and so that the error is approximately equidistributed on the final mesh. The solution is returned on this final mesh.

If the solution is required at a few specific points then these should be included in the initial mesh. If, on the other hand, the solution is required at several specific points, then you should use the interpolation functions provided in Chapter e01 if these points do not themselves form a convenient mesh.

## 4 References

Pereyra V (1979) PASVA3: An adaptive finite-difference Fortran program for first order nonlinear, ordinary boundary problems *Codes for Boundary Value Problems in Ordinary Differential Equations. Lecture Notes in Computer Science* (eds B Childs, M Scott, J W Daniel, E Denman and P Nelson) 76 Springer-Verlag

## 5 Arguments

1: **neq** – Integer *Input*

*On entry:* the number of equations; that is **neq** is the order of system (1).

*Constraint:* **neq**  $\geq$  2.

2: **fcnfn** – function, supplied by the user *External Function*

**fcnfn** must evaluate the matrix  $F(x)$  in (1) at a general point  $x$ .

The specification of **fcnfn** is:

```
void fcnfn (Integer neq, double x, double f[], Nag_User *comm)
```

1: **neq** – Integer *Input*

*On entry:* the number of differential equations.

2: **x** – double *Input*

*On entry:* the value of the independent variable  $x$ .

3: **f[neq  $\times$  neq]** – double *Output*

*On exit:* the  $(i, j)$ th element of the matrix  $F(x)$ , for  $i, j = 1, 2, \dots, \mathbf{neq}$  where  $F_{ij}$  is set by **f**[( $i - 1$ )  $\times$  **neq** + ( $j - 1$ )]. (See Section 9 for an example.)

4: **comm** – Nag\_User \*

Pointer to a structure of type Nag\_User with the following member:

**p** – Pointer

*On entry/exit:* the pointer **comm**  $\rightarrow$  **p** should be cast to the required type, e.g., `struct user *s = (struct user *)comm  $\rightarrow$  p`, to obtain the original object's address with appropriate type. (See the argument **comm** below.)

3: **fcng** – function, supplied by the user *External Function*

**fcng** must evaluate the vector  $g(x)$  in (1) at a general point  $x$ .

The specification of **fcng** is:

```
void fcng (Integer neq, double x, double g[], Nag_User *comm)
```

1: **neq** – Integer *Input*

*On entry:* the number of differential equations.

2: **x** – double *Input*

*On entry:* the value of the independent variable  $x$ .

3:	<b>g[neq]</b> – double	<i>Output</i>
	<i>On exit:</i> the $i$ th element of the vector $g(x)$ , for $i = 1, 2, \dots, \mathbf{neq}$ . (See Section 9 for an example.)	
4:	<b>comm</b> – Nag_User *	
	Pointer to a structure of type Nag_User with the following member:	
	<b>p</b> – Pointer	
	<i>On entry/exit:</i> the pointer <b>comm</b> $\rightarrow$ <b>p</b> should be cast to the required type, e.g., struct user *s = (struct user *)comm $\rightarrow$ p, to obtain the original object's address with appropriate type. (See the argument <b>comm</b> below.)	

If you do not wish to supply **fcng** the actual argument **fcng** must be the NAG defined null function pointer NULLFN.

- 4: **a** – double *Input*  
*On entry:* the left-hand boundary point,  $a$ .
- 5: **b** – double *Input*  
*On entry:* the right-hand boundary point,  $b$ .  
*Constraint:* **b** > **a**.
- 6: **c[neq  $\times$  neq]** – double *Input/Output*  
7: **d[neq  $\times$  neq]** – double *Input/Output*  
8: **gam[neq]** – double *Input/Output*

*On entry:* the arrays **c** and **d** must be set to the matrices  $C$  and  $D$  in (2). **gam** must be set to the vector  $\gamma$  in (2).

*On exit:* the rows of **c** and **d** and the components of **gam** are re-ordered so that the boundary conditions are in the order:

- (i) conditions on  $y(a)$  only;
- (ii) condition involving  $y(a)$  and  $y(b)$ ; and
- (iii) conditions on  $y(b)$  only.

The function will be slightly more efficient if the arrays **c**, **d** and **gam** are ordered in this way before entry, and in this event they will be unchanged on exit.

Note that the boundary conditions must be of boundary value type, that is neither  $C$  nor  $D$  may be identically zero. Note also that the rank of the matrix  $[C, D]$  must be **neq** for the problem to be properly posed. Any violation of these conditions will lead to an error exit.

- 9: **mnp** – Integer *Input*  
*On entry:* the maximum permitted number of mesh points.  
*Constraint:* **mnp**  $\geq$  32.
- 10: **np** – Integer \* *Input/Output*  
*On entry:* determines whether a default or user-supplied initial mesh is used.
- np** = 0  
**np** is set to a default value of 4 and a corresponding equispaced mesh  $\mathbf{x}[0], \mathbf{x}[1], \dots, \mathbf{x}[\mathbf{np} - 1]$  is used.
- np**  $\geq$  4  
You must define an initial mesh using the array **x** as described.
- Constraint:* **np** = 0 or  $4 \leq \mathbf{np} \leq \mathbf{mnp}$ .

*On exit:* the number of points in the final (returned) mesh.

11: **x[mnp]** – double *Input/Output*

*On entry:* if **np** ≥ 4 (see **np** above), the first **np** elements must define an initial mesh. Otherwise the elements of **x** need not be set.

*Constraint:*  $\mathbf{a} = \mathbf{x}[0] < \mathbf{x}[1] < \dots < \mathbf{x}[\mathbf{np} - 1] = \mathbf{b}$  (3)

, for **np** ≥ 4

*On exit:* **x**[0], **x**[1], ..., **x**[**np** - 1] define the final mesh (with the returned value of **np**) satisfying the relation (3).

12: **y[neq × mnp]** – double *Output*

*On exit:* the approximate solution  $z_j(x_i)$  satisfying (4), on the final mesh, that is

$$\mathbf{y}[j - 1][i - 1] = z_j(x_i), i = 1, 2, \dots, \mathbf{np}; j = 1, 2, \dots, \mathbf{neq},$$

where **np** is the number of points in the final mesh.

The remaining columns of **y** are not used.

13: **tol** – double *Input*

*On entry:* a positive absolute error tolerance.

If

$$a = x_1 < x_2 < \dots < x_{\mathbf{np}} = b \quad (4)$$

is the final mesh,  $z_j(x_i)$  is the *j*th component of the approximate solution at  $x_i$ , and  $y_j(x_i)$  is the *j*th component of the true solution of equation (1) (see Section 3) and the boundary conditions, then, except in extreme cases, it is expected that

$$|z_j(x_i) - y_j(x_i)| \leq \mathbf{tol}, i = 1, 2, \dots, \mathbf{np}; j = 1, 2, \dots, \mathbf{neq} \quad (5)$$

*Constraint:* **tol** > 0.0.

14: **comm** – Nag\_User \*

Pointer to a structure of type Nag\_User with the following member:

**p** – Pointer

*On entry/exit:* the pointer **p**, of type Pointer, allows you to communicate information to and from **fcnf** and **fcng**. An object of the required type should be declared, e.g., a structure, and its address assigned to the pointer **p** by means of a cast to Pointer in the calling program, e.g., `comm.p = (Pointer)&s`. The type pointer will be `void *` with a C compiler that defines `void *` and `char *` otherwise.

15: **fail** – NagError \* *Input/Output*

The NAG error argument (see Section 3.6 in the Essential Introduction).

## 6 Error Indicators and Warnings

### NE\_2\_REAL\_ARG\_LE

On entry, **b** = *<value>* while **a** = *<value>*. These arguments must satisfy **b** > **a**.

### NE\_ALLOC\_FAIL

Dynamic memory allocation failed.

**NE\_BOUND\_COND\_COL**

More than **neq** columns of the **neq** by  $2 \times \text{neq}$  matrix  $[C, D]$  are identically zero. i.e., the boundary conditions are rank deficient. The number of non-identically zero columns is  $\langle \text{value} \rangle$ .

**NE\_BOUND\_COND\_LC**

At least one row of the **neq** by  $2 \times \text{neq}$  matrix  $[C, D]$  is a linear combination of the other rows, i.e., the boundary conditions are rank deficient. The index of the first such row is  $\langle \text{value} \rangle$ .

**NE\_BOUND\_COND\_MAT**

One of the matrices  $C$  or  $D$  is identically zero, i.e., the problem is of initial value and not of the boundary type.

**NE\_BOUND\_COND\_NLC**

At least one row of the **neq** by  $2 \times \text{neq}$  matrix  $[C, D]$  is a linear combination of the other rows determined up to a numerical tolerance, i.e., the boundary conditions are rank deficient. The index of first such row is  $\langle \text{value} \rangle$ . There is some doubt as to the rank deficiency of the boundary conditions. However even if the boundary conditions are not rank deficient they are not posed in a suitable form for use with this function. For example, if

$$C = \begin{pmatrix} 1 & 0 \\ 1 & \epsilon \end{pmatrix}, D = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}$$

and  $\epsilon$  is small enough, this error exit is likely to be taken. A better form for the boundary conditions in this case would be

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \gamma = \begin{pmatrix} \gamma_1 \\ \epsilon^{-1}(\gamma_2 - \gamma_1) \end{pmatrix}$$

**NE\_BOUND\_COND\_ROW**

Row  $\langle \text{value} \rangle$  of the array **c** and the corresponding row of array **d** are identically zero, i.e., the boundary conditions are rank deficient.

**NE\_CONV\_MESH**

A finer mesh is required for the accuracy requested; that is **mnp** is not large enough.

**NE\_CONV\_MESH\_INIT**

The Newton iteration failed to converge on the initial mesh. This may be due to the initial mesh having too few points or the initial approximate solution being too inaccurate. Try using `nag_ode_bvp_fd_nonlin_gen` (d02rac).

**NE\_CONV\_ROUNDOFF**

Solution cannot be improved due to roundoff error. Too much accuracy might have been requested.

**NE\_INT\_ARG\_LT**

On entry, **mnp** =  $\langle \text{value} \rangle$ .

Constraint: **mnp**  $\geq 32$ .

On entry, **neq** =  $\langle \text{value} \rangle$ .

Constraint: **neq**  $\geq 2$ .

**NE\_INT\_RANGE\_CONS\_2**

On entry, **np** =  $\langle \text{value} \rangle$  and **mnp** =  $\langle \text{value} \rangle$ . The argument **np** must satisfy either  $4 \leq \text{np} \leq \text{mnp}$  or **np** = 0.

**NE\_INTERNAL\_ERROR**

An internal error has occurred in this function. Check the function call and any array sizes. If the call is correct then please contact NAG for assistance.

**NE\_LF\_B\_MESH**

On entry, the left boundary value **a**, has not been set to  $\mathbf{x}[0]$ :  $\mathbf{a} = \langle value \rangle$ ,  $\mathbf{x}[0] = \langle value \rangle$ .

**NE\_NOT\_STRICTLY\_INCREASING**

The sequence  $\mathbf{x}$  is not strictly increasing:  $\mathbf{x}[\langle value \rangle] = \langle value \rangle$ ,  $\mathbf{x}[\langle value \rangle] = \langle value \rangle$ .

**NE\_REAL\_ARG\_LE**

On entry, **tol** must not be less than or equal to 0.0:  $\mathbf{tol} = \langle value \rangle$ .

**NE\_RT\_B\_MESH**

On entry, the right boundary value **b**, has not been set to  $\mathbf{x}[\mathbf{np} - 1]$ :  $\mathbf{b} = \langle value \rangle$ ,  $\mathbf{x}[\mathbf{np} - 1] = \langle value \rangle$ .

**7 Accuracy**

The solution returned by the function will be accurate to your tolerance as defined by the relation (4) except in extreme circumstances. If too many points are specified in the initial mesh, the solution may be more accurate than requested and the error may not be approximately equidistributed.

**8 Further Comments**

The time taken by the function depends on the difficulty of the problem, the number of mesh points (and meshes) used and the number of deferred corrections.

In the case where you wish to solve a sequence of similar problems, the use of the final mesh from one case is strongly recommended as the initial mesh for the next.

**9 Example**

We solve the problem (written as a first order system)

$$\epsilon y'' + y' = 0$$

with boundary conditions

$$y(0) = 0, y(1) = 1$$

for the cases  $\epsilon = 10^{-1}$  and  $\epsilon = 10^{-2}$  using the default initial mesh in the first case, and the final mesh of the first case as initial mesh for the second (more difficult) case. We give the solution and the error at each mesh point to illustrate the accuracy of the method given the accuracy request  $\mathbf{tol} = 1.0\text{e-}3$ .

**9.1 Program Text**

```
/* nag_ode_bvp_fd_lin_gen (d02gbc) Example Program.
 *
 * Copyright 1992 Numerical Algorithms Group.
 *
 * Mark 3, 1992.
 * Mark 7 revised, 2001.
 * Mark 8 revised, 2004.
 *
 */

#include <nag.h>
#include <nagx04.h>
#include <math.h>
```

```

#include <stdio.h>
#include <nag_stdlib.h>
#include <nagd02.h>

#ifdef __cplusplus
extern "C" {
#endif
static void NAG_CALL fcnf(Integer neq, double x, double f[], Nag_User *comm);
#ifdef __cplusplus
}
#endif

#define NEQ 2
#define MNP 70

#define C(I, J) c[(I) *tdc + J]
#define D(I, J) d[(I) *tdd + J]
#define Y(I, J) y[(I) *tdy + J]

int main(int argc, char *argv[])
{
    FILE      *fpout;

    Integer   exit_status = 0, i, j, mnp, neq, np, tdc, tdd, tdy;
    NagError  fail;
    Nag_User  comm;
    double    a, b, *c = 0, *d = 0, eps, *gam = 0, tol, *x = 0, *y = 0;

    INIT_FAIL(fail);

    /* Check for command-line IO options */
    fpout = nag_example_file_io(argc, argv, "--results", NULL);

    fprintf(fpout, "nag_ode_bvp_fd_lin_gen (d02gbc) Example Program Results\n");

    /* For communication with function fcnf()
     * assign address of eps to comm.p.
     */
    comm.p = (Pointer)

    neq = NEQ;
    mnp = MNP;
    tol = 1.0e-3;
    np = 0;
    a = 0.0;
    b = 1.0;
    if (mnp >= 32 && neq >= 2)
    {
        if (!(c = NAG_ALLOC(NEQ*NEQ, double)) ||
            !(d = NAG_ALLOC(NEQ*NEQ, double)) ||
            !(gam = NAG_ALLOC(NEQ, double)) ||
            !(x = NAG_ALLOC(MNP, double)) ||
            !(y = NAG_ALLOC(NEQ*MNP, double)))
        {
            fprintf(fpout, "Allocation failure\n");
            exit_status = -1;
            goto END;
        }
        tdc = neq;
        tdd = neq;
        tdy = mnp;
    }
    else
    {
        exit_status = 1;
        return exit_status;
    }

    for (i = 0; i < neq; ++i)
    {
        gam[i] = 0.0;
    }
}

```

```

    for (j = 0; j < neq; ++j)
    {
        C(i, j) = 0.0;
        D(i, j) = 0.0;
    }
}
C(0, 0) = 1.0;
D(1, 0) = 1.0;
gam[1] = 1.0;
for (i = 1; i <= 2; ++i)
{
    eps = pow(10.0, (double) -i);
    fprintf(fpout, "\nProblem with epsilon = %7.4f\n", eps);
    /* nag_ode_bvp_fd_lin_gen (d02gbc).
     * Ordinary differential equations solver, for general
     * linear two-point boundary value problems, using a finite
     * difference technique with deferred correction
     */
    nag_ode_bvp_fd_lin_gen(neq, fcnf, NULLFN, a, b, c, d, gam,
                          mnp, &np, x, y, tol, &comm, &fail);
    if (fail.code != NE_NOERROR)
    {
        fprintf(fpout, "Error from nag_ode_bvp_fd_lin_gen (d02gbc).\n%s\n",
                fail.message);
        exit_status = 1;
        goto END;
    }
    fprintf(fpout, "\nApproximate solution on final mesh of %ld points\n",
            np);
    fprintf(fpout, "    X(I)          Y(1,I)\n");
    for (j = 0; j < np; ++j)
        fprintf(fpout, "%9.4f    %9.4f\n", x[j], Y(0, j));
}
END:
if (fpout != stdout) fclose(fpout);
if (c) NAG_FREE(c);
if (d) NAG_FREE(d);
if (gam) NAG_FREE(gam);
if (x) NAG_FREE(x);
if (y) NAG_FREE(y);
return exit_status;
}

static void NAG_CALL fcnf(Integer neq, double x, double f[],
                          Nag_User *comm)
{
#define F(I, J) f[(I) *neq+J]
    double *eps = (double *) comm->p;

    F(0, 0) = 0.0;
    F(0, 1) = 1.0;
    F(1, 0) = 0.0;
    F(1, 1) = -1.0/ *eps;
}

```

## 9.2 Program Data

None.

## 9.3 Program Results

nag\_ode\_bvp\_fd\_lin\_gen (d02gbc) Example Program Results

Problem with epsilon = 0.1000

Approximate solution on final mesh of 15 points

X(I)	Y(1,I)
0.0000	0.0000
0.0278	0.2425

0.0556	0.4263
0.1111	0.6708
0.1667	0.8112
0.2222	0.8917
0.2778	0.9379
0.3333	0.9644
0.4444	0.9883
0.5556	0.9962
0.6667	0.9988
0.7500	0.9995
0.8333	0.9998
0.9167	0.9999
1.0000	1.0000

Problem with epsilon = 0.0100

Approximate solution on final mesh of 49 points

X(I)	Y(1,I)
0.0000	0.0000
0.0009	0.0884
0.0019	0.1690
0.0028	0.2425
0.0037	0.3095
0.0046	0.3706
0.0056	0.4262
0.0065	0.4770
0.0074	0.5232
0.0083	0.5654
0.0093	0.6038
0.0111	0.6708
0.0130	0.7265
0.0148	0.7727
0.0167	0.8111
0.0185	0.8431
0.0204	0.8696
0.0222	0.8916
0.0241	0.9100
0.0259	0.9252
0.0278	0.9378
0.0306	0.9529
0.0333	0.9643
0.0361	0.9730
0.0389	0.9795
0.0417	0.9845
0.0444	0.9883
0.0472	0.9911
0.0500	0.9933
0.0528	0.9949
0.0556	0.9961
0.0648	0.9985
0.0741	0.9994
0.0833	0.9998
0.0926	0.9999
0.1019	1.0000
0.1111	1.0000
0.1389	1.0000
0.1667	1.0000
0.2222	1.0000
0.2778	1.0000
0.3333	1.0000
0.4444	1.0000
0.5556	1.0000
0.6667	1.0000
0.7500	1.0000
0.8333	1.0000
0.9167	1.0000
1.0000	1.0000