# NAG Toolbox

# nag_lapack_dtgsna (f08yl)

## 1 Purpose

nag_lapack_dtgsna (f08yl) estimates condition numbers for specified eigenvalues and/or eigenvectors of a matrix pair in generalized real Schur form.

## 2 Syntax

```
[s, dif, m, info] = nag_lapack_dtgsna(job, howmny, select, a, b, vl, vr, mm, 'n',
n)
```

```
[s, dif, m, info] = f08yl(job, howmny, select, a, b, vl, vr, mm, 'n', n)
```

## 3 Description

nag_lapack_dtgsna (f08yl) estimates condition numbers for specified eigenvalues and/or right eigenvectors of an $n$ by $n$ matrix pair $(S,T)$ in real generalized Schur form. The function actually returns estimates of the reciprocals of the condition numbers in order to avoid possible overflow.

The pair $(S,T)$ are in real generalized Schur form if $S$ is block upper triangular with 1 by 1 and 2 by 2 diagonal blocks and $T$ is upper triangular as returned, for example, by nag_lapack_dgges (f08xa) or nag_lapack_dggesx (f08xb), or nag_lapack_dhgeqz (f08xe) with **job** = 'S'. The diagonal elements, or blocks, define the generalized eigenvalues $(\alpha_i,\beta_i)$, for $i=1,2,\dots,n$, of the pair $(S,T)$ and the eigenvalues are given by

$$\lambda_i = \alpha_i/\beta_i,$$

so that

$$\beta_i S x_i = \alpha_i T x_i \quad \text{or} \quad S x_i = \lambda_i T x_i,$$

where $x_i$ is the corresponding (right) eigenvector.

If $S$ and $T$ are the result of a generalized Schur factorization of a matrix pair $(A,B)$

$$A = QSZ^{\mathrm{T}}, \quad B = QTZ^{\mathrm{T}}$$

then the eigenvalues and condition numbers of the pair $(S,T)$ are the same as those of the pair $(A,B)$.

Let $(\alpha,\beta)\neq(0,0)$ be a simple generalized eigenvalue of $(A,B)$. Then the reciprocal of the condition number of the eigenvalue $\lambda=\alpha/\beta$ is defined as

$$s(\lambda) = \frac{\left(|y^{\mathrm{T}}Ax|^2 + |y^{\mathrm{T}}Bx|^2\right)^{1/2}}{\left(\|x\|_2\|y\|_2\right)},$$

where $x$ and $y$ are the right and left eigenvectors of $(A,B)$ corresponding to $\lambda$. If both $\alpha$ and $\beta$ are zero, then $(A,B)$ is singular and $s(\lambda)=-1$ is returned.

The definition of the reciprocal of the estimated condition number of the right eigenvector $x$ and the left eigenvector $y$ corresponding to the simple eigenvalue $\lambda$ depends upon whether $\lambda$ is a real eigenvalue, or one of a complex conjugate pair.

If the eigenvalue $\lambda$ is real and $U$ and $V$ are orthogonal transformations such that

$$U^{\mathrm{T}}(A,B)V = (S,T) = \begin{pmatrix} \alpha & * \\ 0 & S_{22} \end{pmatrix}\begin{pmatrix} \beta & * \\ 0 & T_{22} \end{pmatrix},$$

where $S_{22}$ and $T_{22}$ are $(n-1)$ by $(n-1)$ matrices, then the reciprocal condition number is given by

$$\mathrm{Dif}(x) \equiv \mathrm{Dif}(y) = \mathrm{Dif}((\alpha,\beta),(S_{22},T_{22})) = \sigma_{\min}(Z),$$

where $\sigma_{\min}(Z)$ denotes the smallest singular value of the $2(n-1)$ by $2(n-1)$ matrix

$$Z = \begin{pmatrix} \alpha \otimes I & -1 \otimes S_{22} \\ \beta \otimes I & -1 \otimes T_{22} \end{pmatrix}$$

and $\otimes$ is the Kronecker product.

If $\lambda$ is part of a complex conjugate pair and $U$ and $V$ are orthogonal transformations such that

$$U^{\mathrm{T}}(A,B)V = (S,T) = \begin{pmatrix} S_{11} & * \\ 0 & S_{22} \end{pmatrix}\begin{pmatrix} T_{11} & * \\ 0 & T_{22} \end{pmatrix},$$

where $S_{11}$ and $T_{11}$ are two by two matrices, $S_{22}$ and $T_{22}$ are $(n-2)$ by $(n-2)$ matrices, and $(S_{11},T_{11})$ corresponds to the complex conjugate eigenvalue pair $\lambda$, $\bar{\lambda}$, then there exist unitary matrices $U_1$ and $V_1$ such that

$$U_1^H S_{11} V_1 = \begin{pmatrix} s_{11} & s_{12} \\ 0 & s_{22} \end{pmatrix} \quad \text{and} \quad U_1^H T_{11} V_1 = \begin{pmatrix} t_{11} & t_{12} \\ 0 & t_{22} \end{pmatrix}.$$

The eigenvalues are given by $\lambda = s_{11}/t_{11}$ and $\bar{\lambda} = s_{22}/t_{22}$. Then the Frobenius norm-based, estimated reciprocal condition number is bounded by

$$\mathrm{Dif}(x) \equiv \mathrm{Dif}(y) \leq \min(d_1, \max(1, |\mathrm{Re}(s_{11})/\,\mathrm{Re}(s_{22})|), d_2)$$

where $\mathrm{Re}(z)$ denotes the real part of $z$, $d_1 = \mathrm{Dif}((s_{11},t_{11}),(s_{22},t_{22})) = \sigma_{\min}(Z_1)$, $Z_1$ is the complex two by two matrix

$$Z_1 = \begin{pmatrix} s_{11} & -s_{22} \\ t_{11} & -t_{22} \end{pmatrix},$$

and $d_2$ is an upper bound on $\mathrm{Dif}((S_{11},T_{11}),(S_{22},T_{22}))$; i.e., an upper bound on $\sigma_{\min}(Z_2)$, where $Z_2$ is the $(2n-2)$ by $(2n-2)$ matrix

$$Z_2 = \begin{pmatrix} S_{11}^T \otimes I & -I \otimes S_{22} \\ T_{11}^T \otimes I & -I \otimes T_{22} \end{pmatrix}.$$

See Sections 2.4.8 and 4.11 of Anderson *et al.* (1999) and KÔgstrÎm and Poromaa (1996) for further details and information.

## 4 References

Anderson E, Bai Z, Bischof C, Blackford S, Demmel J, Dongarra J J, Du Croz J J, Greenbaum A, Hammarling S, McKenney A and Sorensen D (1999) *LAPACK Users' Guide* (3rd Edition) SIAM, Philadelphia http://www.netlib.org/lapack/lug

KÔgstrÎm B and Poromaa P (1996) LAPACK-style algorithms and software for solving the generalized Sylvester equation and estimating the separation between regular matrix pairs *ACM Trans. Math. Software* **22** 78–103

## 5 Parameters

### 5.1 Compulsory Input Parameters

1: **job** – CHARACTER(1)

Indicates whether condition numbers are required for eigenvalues and/or eigenvectors.

**job** = 'E'
Condition numbers for eigenvalues only are computed.

**job** = 'V'
Condition numbers for eigenvectors only are computed.

**job** = 'B'
Condition numbers for both eigenvalues and eigenvectors are computed.

*Constraint*: **job** = 'E', 'V' or 'B'.

2: **howmny** – CHARACTER(1)

Indicates how many condition numbers are to be computed.

**howmny** = 'A'
Condition numbers for all eigenpairs are computed.

**howmny** = 'S'
Condition numbers for selected eigenpairs (as specified by **select**) are computed.

*Constraint*: **howmny** = 'A' or 'S'.

3: **select**(:) – LOGICAL array

The dimension of the array **select** must be at least $\max(1, \mathbf{n})$ if **howmny** = 'S', and at least 1 otherwise

Specifies the eigenpairs for which condition numbers are to be computed if **howmny** = 'S'. To select condition numbers for the eigenpair corresponding to the real eigenvalue $\lambda_j$, **select**$(j)$ must be set *true*. To select condition numbers corresponding to a complex conjugate pair of eigenvalues $\lambda_j$ and $\lambda_{j+1}$, **select**$(j)$ and/or **select**$(j+1)$ must be set to *true*.

If **howmny** = 'A', **select** is not referenced.

4: **a**$(lda, :)$ – REAL (KIND=nag_wp) array

The first dimension of the array **a** must be at least $\max(1, \mathbf{n})$.

The second dimension of the array **a** must be at least $\max(1, \mathbf{n})$.

The upper quasi-triangular matrix $S$.

5: **b**$(ldb, :)$ – REAL (KIND=nag_wp) array

The first dimension of the array **b** must be at least $\max(1, \mathbf{n})$.

The second dimension of the array **b** must be at least $\max(1, \mathbf{n})$.

The upper triangular matrix $T$.

6: **vl**$(ldvl, :)$ – REAL (KIND=nag_wp) array

The first dimension, $ldvl$, of the array **vl** must satisfy

if **job** = 'E' or 'B', $ldvl \geq \max(1, \mathbf{n})$;
otherwise $ldvl \geq 1$.

The second dimension of the array **vl** must be at least $\max(1, \mathbf{mm})$ if **job** = 'E' or 'B', and at least 1 otherwise.

If **job** = 'E' or 'B', **vl** must contain left eigenvectors of $(S, T)$, corresponding to the eigenpairs specified by **howmny** and **select**. The eigenvectors must be stored in consecutive columns of **vl**, as returned by nag_lapack_dggev (f08wa) or nag_lapack_dtgevc (f08yk).

If **job** = 'V', **vl** is not referenced.

7: **vr**$(ldvr, :)$ – REAL (KIND=nag_wp) array

The first dimension, $ldvr$, of the array **vr** must satisfy

if **job** = 'E' or 'B', $ldvr \geq \max(1, \mathbf{n})$;
otherwise $ldvr \geq 1$.

The second dimension of the array **vr** must be at least $\max(1, \textbf{mm})$ if **job** = 'E' or 'B', and at least 1 otherwise.

If **job** = 'E' or 'B', **vr** must contain right eigenvectors of $(S, T)$, corresponding to the eigenpairs specified by **howmny** and **select**. The eigenvectors must be stored in consecutive columns of **vr**, as returned by nag_lapack_dggev (f08wa) or nag_lapack_dtgevc (f08yk).

If **job** = 'V', **vr** is not referenced.

8: **mm** – INTEGER

The number of elements in the arrays **s** and **dif**.

*Constraints*:

if **howmny** = 'A', **mm** $\geq$ **n**;
otherwise **mm** $\geq$ **m**.

## 5.2 Optional Input Parameters

1: **n** – INTEGER

*Default*: the first dimension of the arrays **a**, **b** and the second dimension of the arrays **a**, **b**. (An error is raised if these dimensions are not equal.)

$n$, the order of the matrix pair $(S, T)$.

*Constraint*: **n** $\geq 0$.

## 5.3 Output Parameters

1: **s**(:) – REAL (KIND=nag_wp) array

The dimension of the array **s** will be $\max(1, \textbf{mm})$ if **job** = 'E' or 'B' and 1 otherwise

If **job** = 'E' or 'B', the reciprocal condition numbers of the selected eigenvalues, stored in consecutive elements of the array. For a complex conjugate pair of eigenvalues two consecutive elements of **s** are set to the same value. Thus **s**$(j)$, **dif**$(j)$, and the $j$th columns of VL and VR all correspond to the same eigenpair (but not in general the $j$th eigenpair, unless all eigenpairs are selected).

If **job** = 'V', **s** is not referenced.

2: **dif**(:) – REAL (KIND=nag_wp) array

The dimension of the array **dif** will be $\max(1, \textbf{mm})$ if **job** = 'V' or 'B' and 1 otherwise

If **job** = 'V' or 'B', the estimated reciprocal condition numbers of the selected eigenvectors, stored in consecutive elements of the array. For a complex eigenvector two consecutive elements of **dif** are set to the same value. If the eigenvalues cannot be reordered to compute **dif**$(j)$, **dif**$(j)$ is set to 0; this can only occur when the true value would be very small anyway.

If **job** = 'E', **dif** is not referenced.

3: **m** – INTEGER

The number of elements of the arrays **s** and **dif** used to store the specified condition numbers; for each selected real eigenvalue one element is used, and for each selected complex conjugate pair of eigenvalues, two elements are used. If **howmny** = 'A', **m** is set to **n**.

4: **info** – INTEGER

**info** = 0 unless the function detects an error (see Section 6).

## 6 Error Indicators and Warnings

**info** $= -i$

If **info** $= -i$, parameter $i$ had an illegal value on entry. The parameters are numbered as follows:

1: **job**, 2: **howmny**, 3: **select**, 4: **n**, 5: **a**, 6: **lda**, 7: **b**, 8: **ldb**, 9: **vl**, 10: **ldvl**, 11: **vr**, 12: **ldvr**, 13: **s**, 14: **dif**, 15: **mm**, 16: **m**, 17: **work**, 18: **lwork**, 19: **iwork**, 20: **info**.

It is possible that **info** refers to a parameter that is omitted from the MATLAB interface. This usually indicates that an error in one of the other input parameters has caused an incorrect value to be inferred.

## 7 Accuracy

None.

## 8 Further Comments

An approximate asymptotic error bound on the chordal distance between the computed eigenvalue $\tilde{\lambda}$ and the corresponding exact eigenvalue $\lambda$ is

$$\chi\left(\tilde{\lambda},\lambda\right) \leq \epsilon\|(A,B)\|_F/S(\lambda)$$

where $\epsilon$ is the ***machine precision***.

An approximate asymptotic error bound for the right or left computed eigenvectors $\tilde{x}$ or $\tilde{y}$ corresponding to the right and left eigenvectors $x$ and $y$ is given by

$$\theta(\tilde{z},z) \leq \epsilon\|(A,B)\|_F/\mathrm{Dif}.$$

The complex analogue of this function is nag_lapack_ztgsna (f08yy).

## 9 Example

This example estimates condition numbers and approximate error estimates for all the eigenvalues and eigenvalues and right eigenvectors of the pair $(S,T)$ given by

$$S = \begin{pmatrix} 4.0 & 1.0 & 1.0 & 2.0 \\ 0 & 3.0 & -1.0 & 1.0 \\ 0 & 1.0 & 3.0 & 1.0 \\ 0 & 0 & 0 & 6.0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 2.0 & 1.0 & 1.0 & 3.0 \\ 0 & 1.0 & 0.0 & 1.0 \\ 0 & 0 & 1.0 & 1.0 \\ 0 & 0 & 0 & 2.0 \end{pmatrix}.$$

The eigenvalues and eigenvectors are computed by calling nag_lapack_dtgevc (f08yk).

### 9.1 Program Text

```
     function f08yl_example

fprintf('f08yl example results\n\n');

% Generalized Schur form pair (S,T)
  n = nag_int(4);
S = [4, 1,  1, 2;
     0, 3, -1, 1;
     0, 1,  3, 1;
     0, 0,  0, 6];
T = [2, 1,  1, 3;
     0, 1,  0, 1;
     0, 0,  1, 1;
     0, 0,  0, 2];

% Obtain scaled eigenvectors from Schur form
job = 'Both';
howmny = 'All';
```

```
select = [false];
Q = eye(n);
Z = Q;
[VL, VR, m, info] = f08yk( ...
                           job, howmny, select, S, T, Q, Z, n);

% Estimate condition numbers for eigenvalues and right eigenvectors
[rconde, rcondv, m, info] = ...
  f08yl ( ...
          job, howmny, select, S, T, VL, VR, n);

disp('Reciprocal condition numbers for eigenvalues of (S,T)');
fprintf('%11.1e',rconde);
fprintf('\n\n');
disp('Reciprocal condition numbers for right eigenvectors of (S,T)');
fprintf('%11.1e',rcondv);
fprintf('\n\n');

% Calculate approximate error estimates

snorm = norm(S,1);
tnorm = norm(T,1);
stnorm = sqrt(snorm^2 + tnorm^2);

disp('Approximate error estimates for eigenvalues of (S,T)')
erre = x02aj*stnorm./rconde;
fprintf('%11.1e',erre);
fprintf('\n\n');
disp('Approximate error estimates for right eigenvectors of (S,T)')
errv = x02aj*stnorm./rcondv;
fprintf('%11.1e',errv);
fprintf('\n');
```

## 9.2 Program Results

```
     f08yl example results

Reciprocal condition numbers for eigenvalues of (S,T)
    1.6e+00    1.7e+00    1.7e+00    1.4e+00

Reciprocal condition numbers for right eigenvectors of (S,T)
    5.4e-01    1.5e-01    1.5e-01    1.2e-01

Approximate error estimates for eigenvalues of (S,T)
    8.7e-16    7.8e-16    7.8e-16    9.9e-16

Approximate error estimates for right eigenvectors of (S,T)
    2.5e-15    9.0e-15    9.0e-15    1.1e-14
```