Locally Stationary Wavelet Processes with Trend

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Modelling Nonstationary Time Series

Time series are often nonstationary: the mean and/or autocovariance vary over time. We wish to estimate the:

- spectrum/autocovariance: the dependency structure in the time series;
- trend: the smooth, longer-term behaviour of the time series.

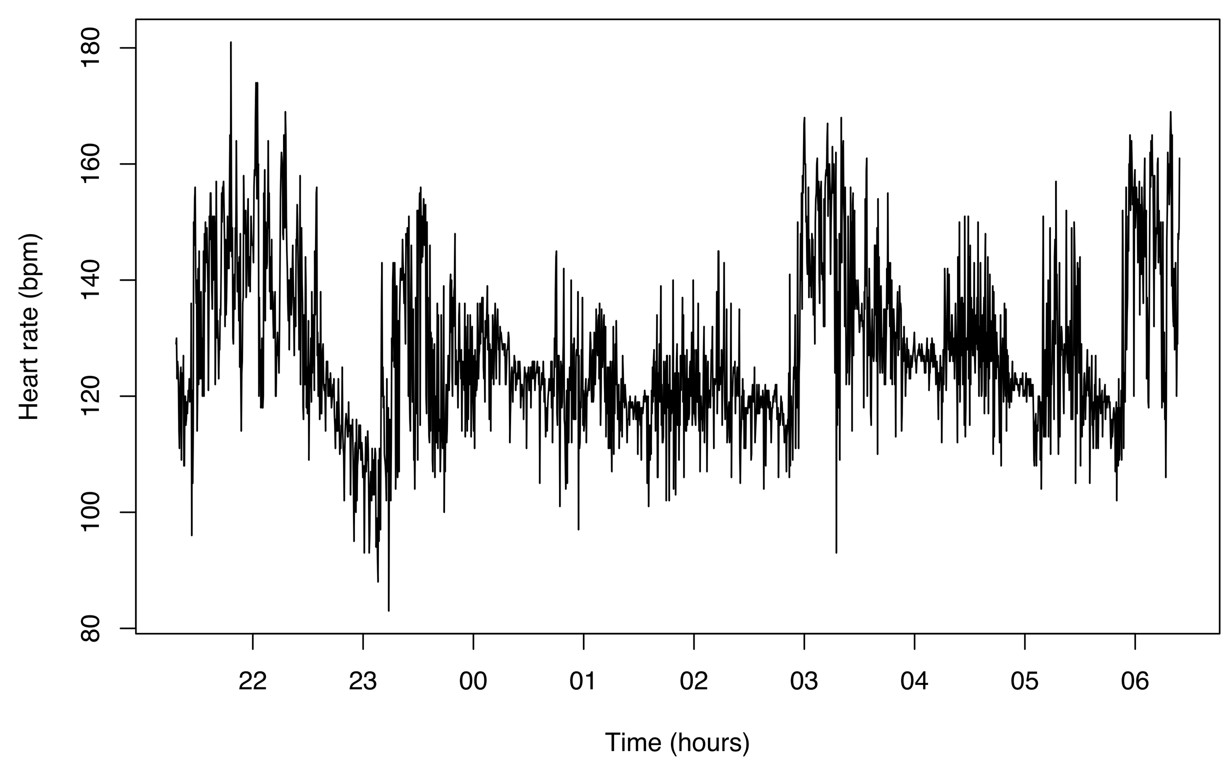

Figure 1: Electrocardiogram (ECG) recording of an infant, courtesy of the Institute of Child Health at the University of Bristol, available in the `wavethresh` R package.

Trend + LSW Model

A nonstationary time series can be modelled as a trend + locally stationary wavelet (LSW) process $\{X_{t,T}\}$, $t = 0, \dots, T-1$, $T = 2^J$, as follows:

$$X_{t,T} = \mu\left(\frac{t}{T}\right) + \sum_{j=-J}^{-1} \sum_{k \in \mathbb{Z}} W_j\left(\frac{k}{T}\right) \psi_{j,k-t}\xi_{j,k},$$

where:

- $\mu(t/T)$ is the trend function, assumed to be Lipschitz continuous.
- $W_j(k/T)$ are amplitudes whose smoothness controls the nonstationarity.
- $\{\psi_{j,k-t}\}_{j,k}$ is a set of discrete non-decimated wavelets.
- $\{\xi_{j,k}\}$ is a sequence of zero-mean, orthonormal random variables.

The no trend case is covered in Nason et al. (2000). We provide rigorous theory for estimating first and second order structure in the Lipschitz case.

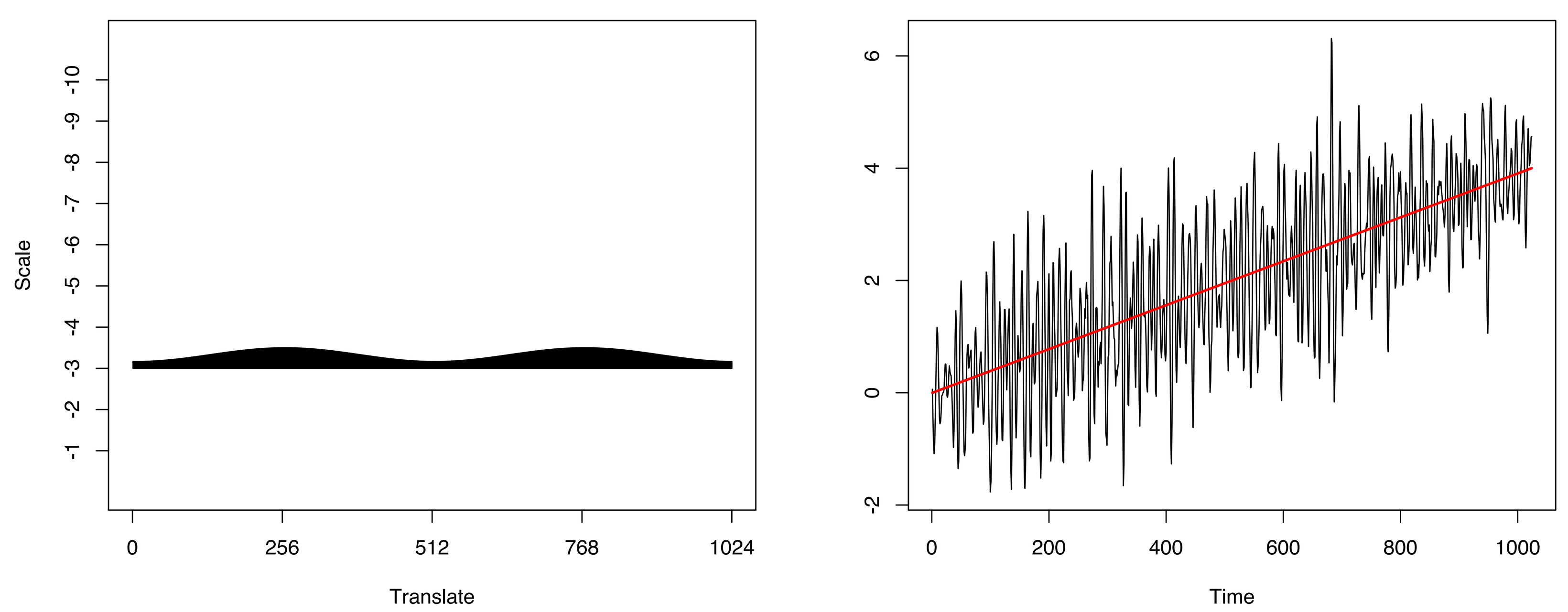

Figure 2: Example trend + LSW process. Left: spectrum which displays slowly-evolving autocovariance. Right: realisation of the process, which has a linear trend (red).

The Wavelet Spectrum

The wavelet spectrum is defined as:

$$S_j(z) = |W_j(z)|^2, \text{ for } z = k/T \in (0,1).$$

- It measures the contribution to variance at a particular rescaled time z and scale j.
- We estimate this quantity using non-decimated wavelet coefficients.
- We correct and smooth to obtain an unbiased, consistent estimate.
- The autocovariance estimate is derived from the spectral estimate.

Differencing the Time Series

- First differencing removes a Lipschitz trend from a time series, and hence from the wavelet coefficients.
- This will alter the autocovariance structure.
- We must modify the standard spectral estimation procedure.
- Lipschitz assumption can be relaxed: a higher order difference is required.

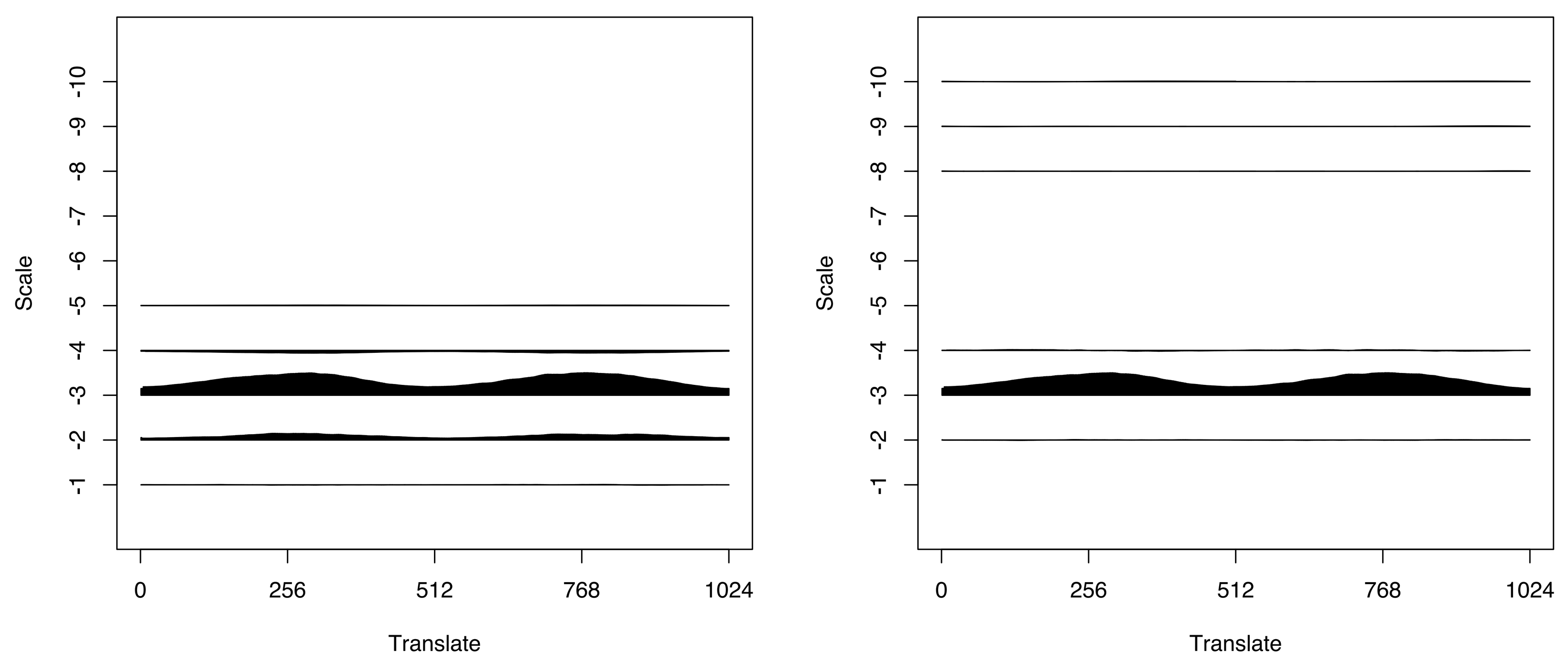

Figure 3: Left: mean of 100 uncorrected smoothed spectral estimates of the process from Figure 2. Right: mean of 100 corrected smoothed estimates.

Spectral Estimation Procedure

- Use wavelet coefficients of the differenced time series.
- Obtain unbiased spectral estimate by modifying the correction applied.
- Analogous theoretical results from the no trend case can be obtained.
- Consistency of spectrum and autocovariance likewise follow.

References

- Nason, G. P., Von Sachs, R., and Kroisandt, G. (2000). Wavelet processes and adaptive estimation of the evolutionary wavelet spectrum. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 62(2):271-292.
- Fryzlewicz, P. and Nason, G. P. (2006). Haar-Fisz estimation of evolutionary wavelet spectra. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 68(4):611-634.
- Nason, G. (2010). wavethresh: Wavelets statistics and transforms. R package version, 4(6.2013).

Trend Estimation Procedure

We can take two approaches:

1. Decorrelate the original time series using the Mahalanobis whitening matrix $W = \widehat{\Sigma}^{-1/2}$, where $\widehat{\Sigma}$ is the variance-covariance matrix estimate.

   Smooth the transformed, decorrelated time series using standard methods such as wavelet thresholding or splines.

   Invert the transformed series using W^{-1} to obtain trend estimate.
2. Perform trend estimation on the original series, e.g. with wavelet thresholding, using the autocovariance estimate to choose the threshold.

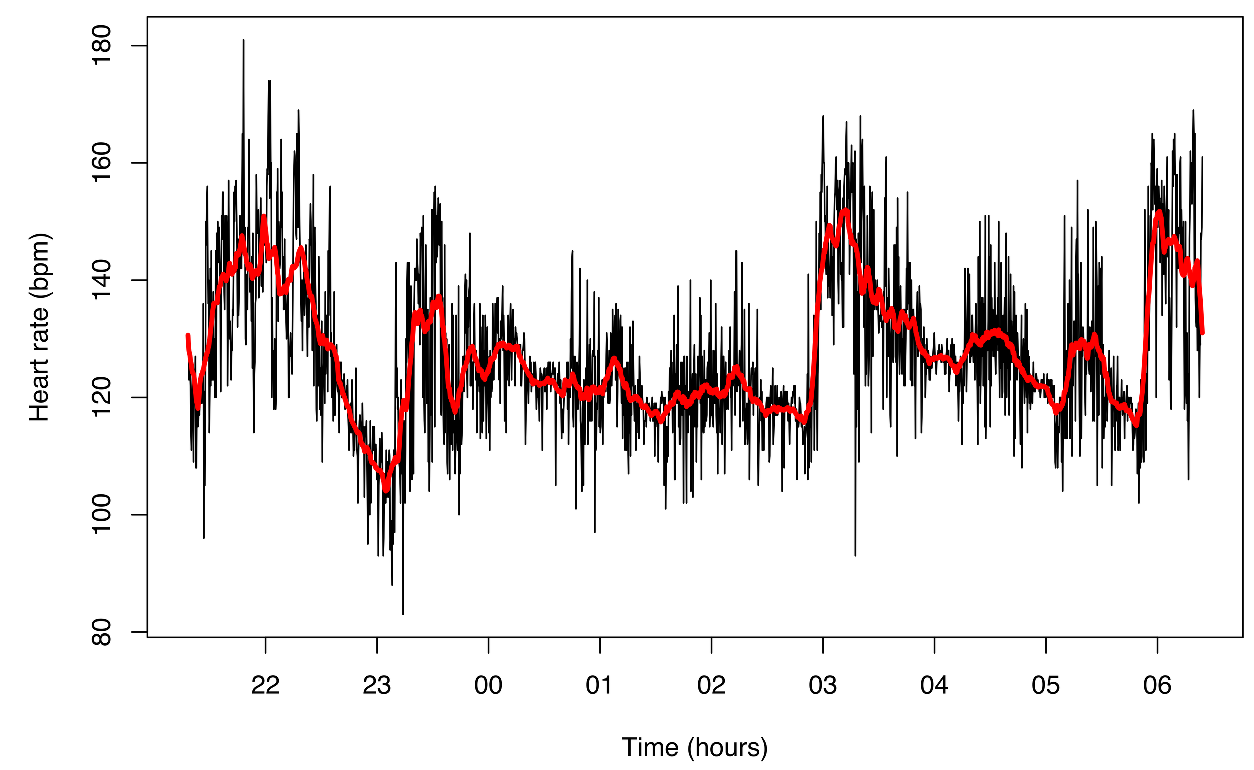

Figure 4: Estimate of the trend (red) of the ECG data from Figure 1, obtained via method 2.

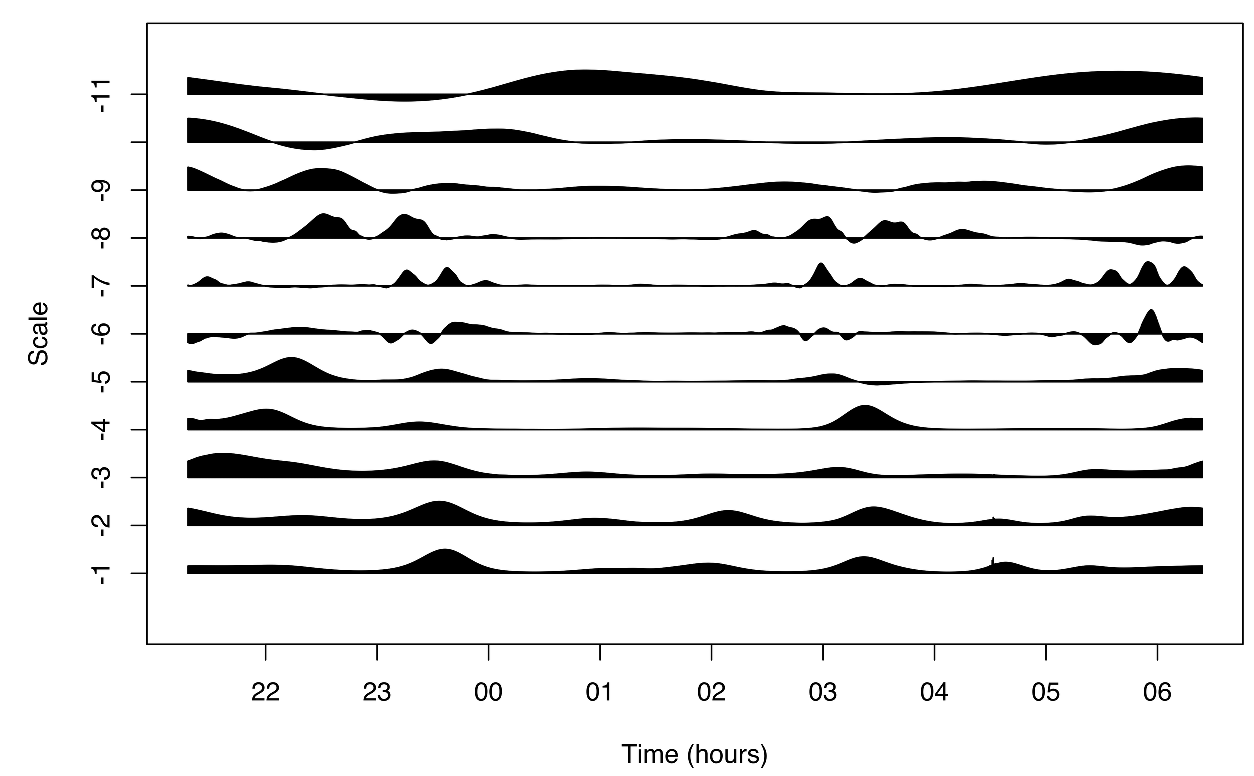

Figure 5: Estimate of the spectrum of the ECG data from Figure 1, using the Daubechies least asymmetric wavelet with 10 vanishing moments. Spectrum smoothed using wavelet thresholding. Each level in the plot is scaled independently for clarity.

Conclusions/Future Work

- Given a time series that exhibits a smooth trend and nonstationary autocovariance, we can estimate both quantities within this framework.
- Future work: perform wide-ranging simulation study on the method.
- Extend to non-smooth trends, e.g. piecewise constants.
- Develop methodology to statistically test for the presence of trend.

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