Numerical Algorithms Group

Mathematics and technology for optimized performance

Software Issues in Wavelet Analysis of Financial Data

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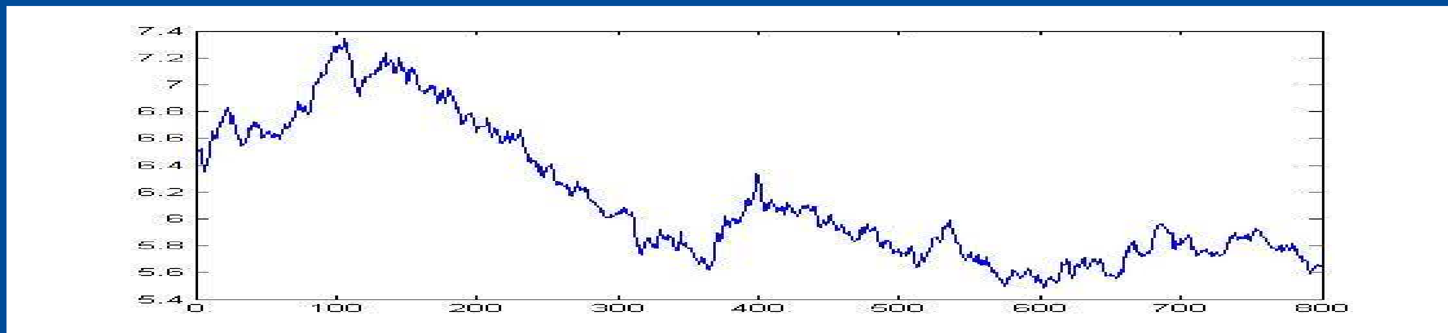
Results Matter. Trust NAG.

Overview

- *Why Use Wavelets?*
- *Wavelet transforms*
- *Multi-Resolution Analysis*
- *Software implementations and algorithms*
- *Choosing a wavelet method*
- *Some Applications*

Why Use Wavelets?



Signal → stream of data in time

To analyse structure of time series:

- **DFT** (Discrete Fourier Transform) → frequency representation
- **STFT** (Short Time Fourier Transform, Gabor) → uses a time window to give localisation in time – imposes a scale which leads to aliasing of components
- **Wavelet** Transform → shifted and scaled basis functions allow localisation in time and frequency
- **Uncertainty principle**: cannot achieve simultaneous time and frequency resolution

Wavelet Transforms

Decompose time series, $x(t)$, by convolution with dilated and translated mother wavelet, $\psi(t)$

- Continuous (CWT)

$$d(u,s) = \int_{-\infty}^{\infty} x(t)\psi_{u,s}(t)\,dt,$$

$$\psi_{u,s}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-u}{s}\right)$$

e.g. Morlet wavelet,

$$\psi(t) = \frac{1}{\sqrt{2\pi}} e^{-ikt} e^{-t^2/2}$$

- Discrete (DWT)

Filter pair: G – high pass
H – low pass
with D – down-sampling

$$(Hx)_k = \sum_n h_{n-k} x_n$$

Wavelet Transforms (2)

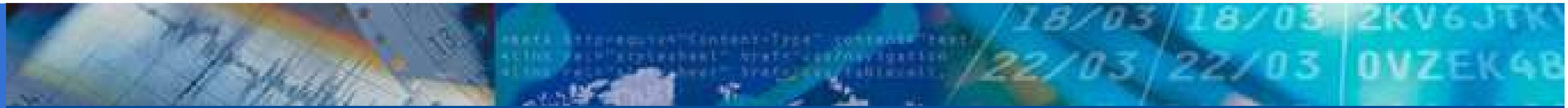
- **CWT requires:**

$$\int_{-\infty}^{\infty} \psi(t)dt = 0 \qquad \int_{-\infty}^{\infty} |\psi(t)|^2 \, dt = 1$$

- **DWT (orthogonal filter pair) requires:**

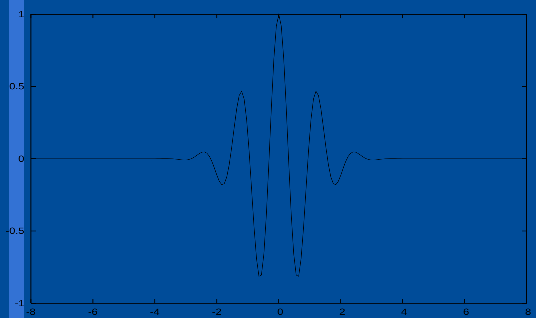
$$\sum_n h_n h_{n+2j} = 0, \qquad \sum_n h_n^2 = 1, \qquad g_n = (-1)^n h_{1-n}$$

$$\sum_n g_n g_{n+2j} = 0, \qquad \sum_n g_n^2 = 1, \qquad \sum_n h_n g_{n+2j} = 0$$



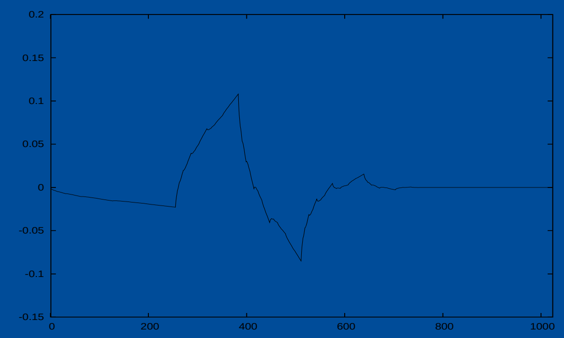
Wavelet Functions

Morlet (k=5)

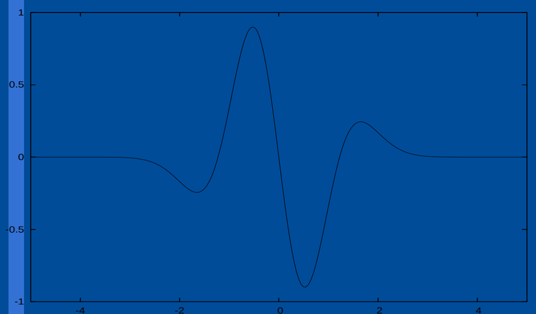


Daubechies:

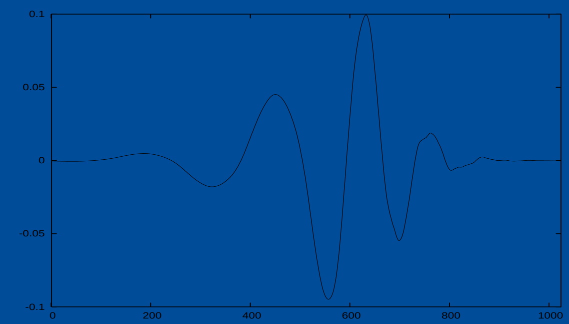
DB4



Gaussian 3rd derivative

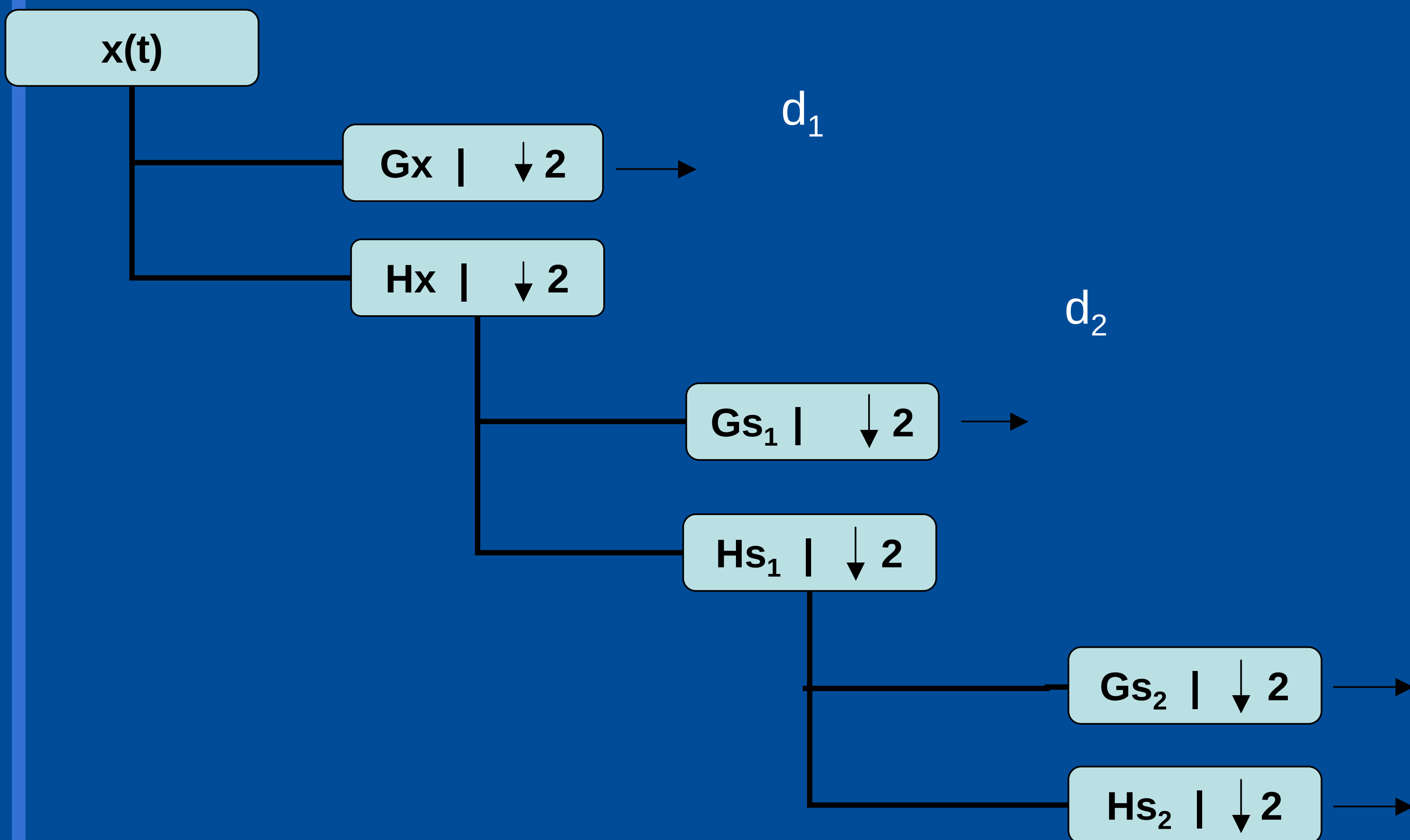


DB12



Multi-Resolution Analysis

Discrete Wavelet Transform



Relation between *Continuous* and *Discrete* Transforms

- Scaling function is fixed point of *H*:

$$\varphi = H\varphi ,$$

$$\varphi(t) = \sqrt{2}\sum_j h_j \varphi(2t - j)$$

- Mother wavelet:

$$\psi(t) = \sqrt{2}\sum_j g_j \varphi(2t - j)$$

- Time series:

(see Strang & Nguyen, 1997)

$$x(t) = \sum_k c_k^J \varphi_J(t - 2^{-J}k)$$

- CWT detail coefficients:

$$x(t) = \sum_k c_k^R \varphi_R(t - 2^{-R}k) + \sum_{j=R}^{J-1} d_k^j \psi_j(t - 2^{-j}k)$$

$$d_k^j = \int \psi_j(t - 2^{-j}k)\, x(t)\, dt$$

Stationary DWT (SDWT)

- Note: DWT is **NOT** translation invariant
  *choosing **odd** entries in series, in place of **even** ones when down-sampling gives a different orthogonal transformation*
- DWT MRA – *choice of shift at each level gives multiple possible sets of coefficients*
- SDWT – ***NO** down-sampling,*
  pad filters with zeros in MRA
  includes all DWT MRA possibilities
  translation invariant, but increases storage
  can relate wavelet coefficients to data

Translation invariant SDWT

$x(t)$

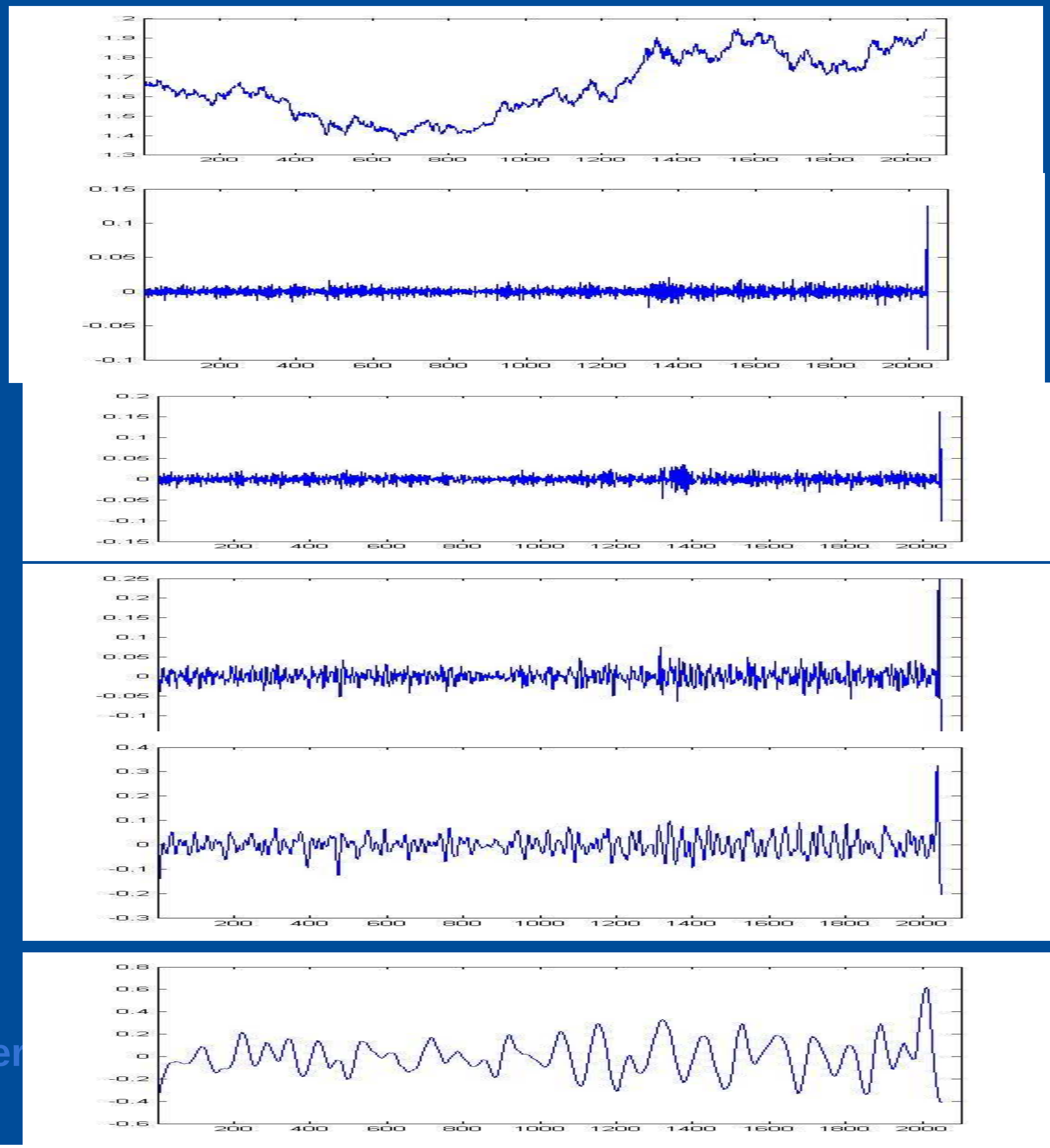
d_1

d_2

d_3

d_4

s_4



Choosing a Wavelet Method

- CWT

  Continuous transform

  Visualise as surface

- DWT/MRA

  Discrete, multiresolution

  Efficient storage of signal

- Matching Pursuit

  Adapt basis to data

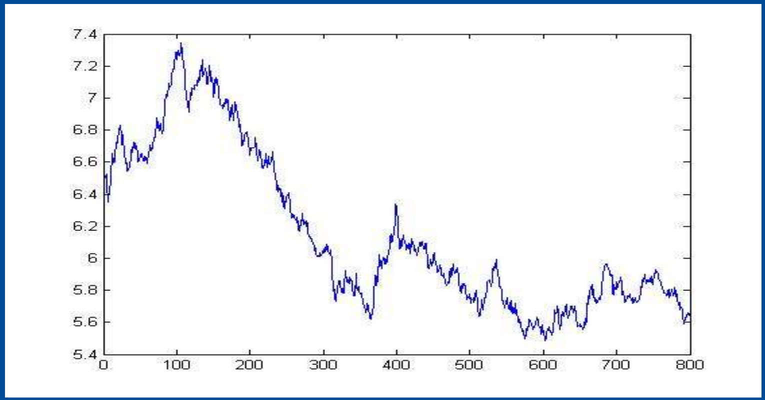
  At each level of MRA choose waveform to minimise residual passed to next level

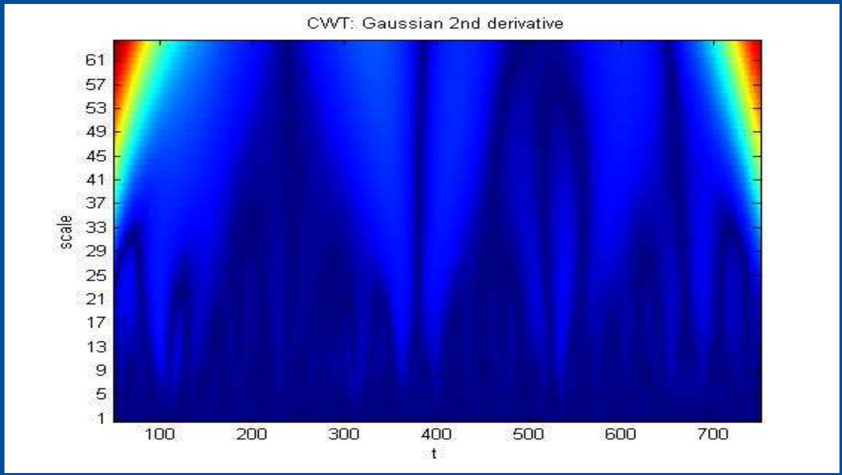
- SDWT

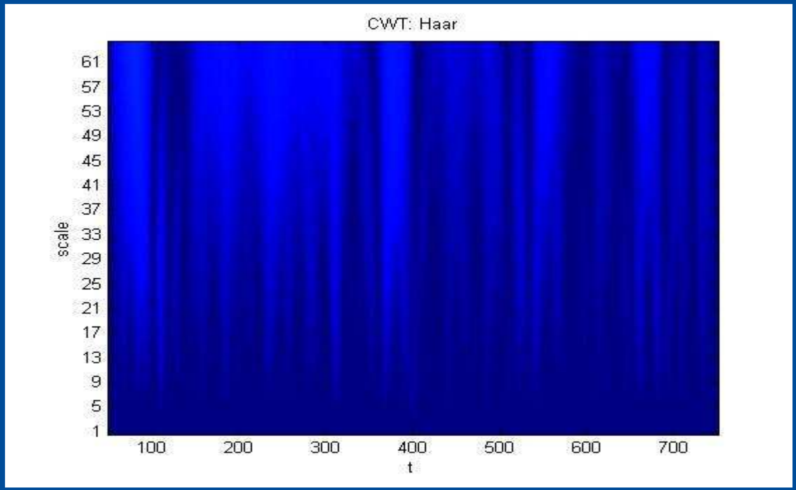
  Translation invariant

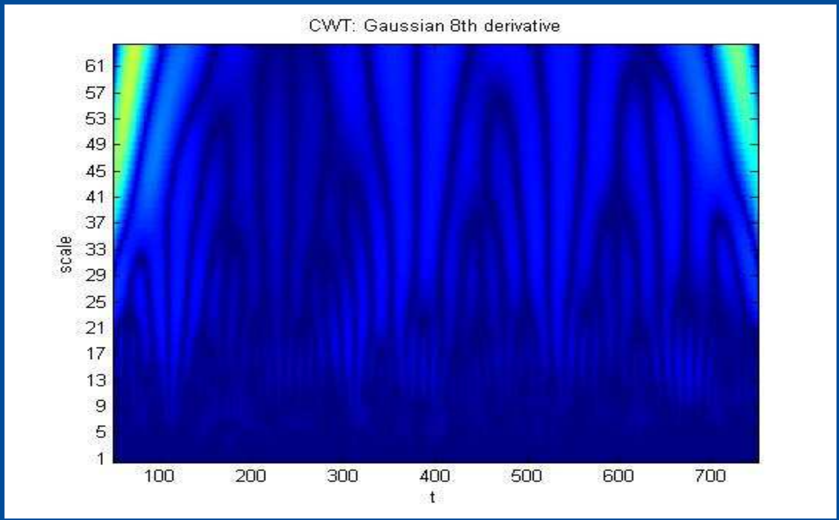
  No down-sampling

CWT: how can quantitative information be obtained?









Results Matter. Trust NAG.

CWT (2)

- Find common normalisation for wavelet spectrum – ensure wavelet has unit energy at each scale
- Choice of wavelet:

  non-orthogonal is useful for time series, but highly redundant
- Choice of scales:

  can use arbitrary set of scales to show structure
- Cone of influence:

  for finite length series defines where edge effects occur
- Relate to Fourier frequency

e.g. see Torrence and Compo (1998)

Software implementation and algorithms

- Reproducibility is desirable –

  algorithms precisely defined to allow independent implementations – Taswell (1998), *c.f.* Buckheit and Donoho (1995)
- Edge effects –

  contaminate ends of transform for finite signals – various end conditions used to reduce their effect: periodic extension, reflection, zero-padding …
- CWT –

  implement quadrature and convolution in time domain or

  else convolution with Fourier Transform of wavelet in frequency domain
- Parallel implementation

Implementation and algorithms (2)

- Definition of forward and inverse transforms –
  DWT orthogonality conditions allow for different choices of forward and inverse transforms
- Pre-processing of data –
  data may need cleaning, interpolation to produce homogeneous series, …

Applications

- De-noising
- Identifying seasonality
- Self-similarity
- Prediction
- Estimation of variance

De-Noising

- Transform data into wavelet domain
- Apply thresholding – suppress *smallest* coefficients
- Transform back

Use:

DWT for efficient storage – SDWT to align with data

De-noised data can help modelling of underlying structure

e.g. Capobianco (1997) – analysis of Nikkei index

Donoho and Johnstone (1998)

Identifying Seasonality

- Apply SDWT to data
- Wavelet detail coefficients at a given level capture a particular range of frequencies
- Identify detail coefficients carrying seasonal periodicity
- Filter out seasonal effects

e.g. Gencay *et al.* (2001) for application to FX returns

Self-similarity

- Scalogram represents energy of series in the wavelet coefficients

e.g. Jamdee and Los (2004) use Morlet wavelet CWT for analysis of interest rate series and calculation of Hurst exponent

Prediction

- Apply backward looking wavelet MRA to time series
- Use wavelet coefficients as input to a neural network model for prediction

e.g. Renaud *et al.* (2004)

Estimation of Variance

For DWT coefficients *w*

SDWT coefficients w^s

$$\| x \|^2 = \| w \|^2 = \| w^s \|^2$$

Since,

$$\| x \|^2 \propto Var(x)$$

The wavelet transform provides an alternative representation of the variance (Percival, 1995)

Summary

1) Wavelet transforms provide a rigorous framework for data analysis in time and frequency
2) Software implementations vary in their choices of transform definitions
3) Wavelet analysis provides an important tool for determining the structure of time series arising in finance